Parametric Statistics

Hypothesis Testing

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Lecture Summary

Example: Power Function

- Testing hypotheses: $X_1, \ldots, X_n \sim Bern(p)$.
- $H_0 : p \leq 0.3$ vs $H_1 : p > 0.3$.

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- Statistic $Y = \sum X_i$, if $R : Y \in (np_0 + c, \infty)$
- Power function:

$$\pi(p|\delta) = P(Y > np_0 + c|p)$$

- We can compute this since $Y \sim Binom(n, p)$
- Power function is increasing in p, supremum for $p \in \Omega_0$ for $p = 0.3$.

The Likelihood Ratio test

Likelihood Ratio Test

The statistic

$$\Lambda(\boldsymbol{x}) = \frac{\sup_{\theta \in \Omega_0} f_n(\boldsymbol{x} \mid \theta)}{\sup_{\theta \in \Omega} f_n(\boldsymbol{x} \mid \theta)}$$

is called the likelihood ratio statistic. A likelihood ratio test of hypotheses is to reject H_0 if $\Lambda(\boldsymbol{x}) \leq k$ for some constant k.

Example

- $X_1, \ldots, X_n \sim Bern(p)$
- $H_0 : p = 0.5$ vs $H_1 : p \neq 0.5$.
- Find the likelihood ratio statistic and find a test with level 0.05.

Large Sample Likelihood Ratio Test

Theorem

Let Ω be an open subset of p-dimensional space, and suppose that H_0 specifies that k coordinates of θ are equal to k specific values. Assume that H_0 is true and that the likelihood function satisfies the conditions needed to prove that the M.L.E. is asymptotically normal and asymptotically efficient. Then, as $n \to \infty$, $-2 \log \Lambda(\boldsymbol{X})$ converges in distribution to the χ^2 distribution with k degrees of freedom.

Uniformly Most Powerful Tests

$$H_0 : \theta \in \Omega_0 \quad \text{vs} \quad H_1 : \theta \in \Omega_1$$

- A test δ^* is a uniformly most powerful test at level α_0 if for any other level α_0 test δ

$$\pi(\theta \mid \delta) \leq \pi\left(\theta \mid \delta^*\right) \quad \text{for all } \theta \in \Omega_1$$

It has the lowest probability of type II error of any test, uniformly for all $\theta \in \Omega_1$.

- We control the probability of type I error by setting the level (size) of the test low. We then want to control the probability of type II error.
- If $\pi\left(\theta \mid \delta^*\right)$ is high for all $\theta \in \Omega_1$, the test is often called "powerful"
- In a large class of problems (the distribution has a "monotone likelihood ratio") we can find a uniformly most powerful test for one-sided hypotheses (Ch. 9.3).

Hypothesis tests vs Confidence Intervals

Rain from Seeded Clouds

- Without seeding: $\mu = 4$.
- With seeding: $\mu = 5.136$.
- $H_0 : \mu \leq 4 \quad H_1 : \mu > 4$.
- $n = 26, \sigma = 1.6$.

- Find a 0.05-level test for H_0.
- Find a 95% confidence interval for μ.

Hypothesis tests vs Confidence Intervals

Theorem

Suppose that for every value θ_0 in Θ there is a test at level α of the hypothesis $H_0 : \theta = \theta_0$. Denote the rejection region of the test by $R(\theta_0)$. Then the set

$$C(\mathbf{X}) = \{\theta : \mathbf{X} \notin R(\theta)\}$$

is a $100(1-\alpha)\%$ confidence interval for θ.

Suppose that $C(\mathbf{X})$ is a $100(1-\alpha)\%$ confidence interval for θ; that is, for every θ_0,

$$P\left[\theta_0 \in C(\mathbf{X})\right] \theta = \theta_0] = 1 - \alpha$$

Then a rejection for a test at level α of the hypothesis $H_0 : \theta = \theta_0$ is

$$A(\theta_0) = \{\mathbf{X} \mid \theta_0 \notin C(\mathbf{X})\}$$

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- For a test with level α_0,

$$sup_{p \in \Omega_0} \pi(p|\delta) = P(Y > np_0 + c|p = p_0) =$$

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$$

- If, additionally, $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$.

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- If, additionally, $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$.
- Then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\hat{\sigma}} \xrightarrow{d} \mathcal{N}(0, 1)$$