Parametric Statistics-Recitation 8 (Solutions)

Instructions for Exercises 1 to 5 : In each of these exercises, assume that the random variables $X_1, \ldots, X_n$ form a random sample of size n from the distribution specified in that exercise, and show that the statistic T specified in the exercise is a sufficient statistic for the parameter.

Exercise 1.

The Bernoulli distribution with parameter p, which is unknown $(0 < p < 1)$; $T = \sum_{i=1}^{n} X_i$.

Solution.

The joint p.d.f is

$$f(\mathbf{x}|p) = p^T(1-p)^{n-T}$$

So the Factorization theorem works for

$$u(\mathbf{x}) = 1 \text{ and } v(T,p) = p^T(1-p)^{n-T}$$

Exercise 2.

The geometric distribution with parameter p, which is unknown $(0 < p < 1)$; $T = \sum_{i=1}^{n} X_i$.

Solution.

The joint p.d.f is

$$f(\mathbf{x}|p) = p^n(1-p)^T$$

So the Factorization theorem works for

$$u(\mathbf{x}) = 1 \text{ and } v(T,p) = p^n(1-p)^T$$

Exercise 3.

The normal distribution for which the mean μ is known and the variance $\sigma^2 > 0$ is unknown; $T = \sum_{i=1}^{n} (X_i - \mu)^2$.

Solution.

The joint p.d.f is

$$f(\mathbf{x}|\sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{\frac{-T}{2\sigma^2}}$$

So the Factorization theorem works for

$$u(\mathbf{x}) = 1 \text{ and } v(T,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{\frac{-T}{2\sigma^2}}$$

Exercise 4.

The gamma distribution with parameters α and β, where the value of α is known and the value of β is unknown $(\beta > 0)$; $T = \bar{X}_n$.

Solution.

The joint p.d.f is

$$f(\mathbf{x}|\beta) = \frac{1}{(\Gamma(a))^n}\left[\prod_{i=1}^{n} x_i\right]^{a-1} \beta^{na} e^{-n\beta T}$$

So the Factorization theorem works for

$$u(\mathbf{x}) = \frac{1}{(\Gamma(\alpha))^n}\left[\prod_{i=1}^{n} x_i\right]^{\alpha-1} \text{ and } v(T,\beta) = \beta^{n\alpha} e^{-n\beta T}$$

Exercise 5.

The gamma distribution with parameters α and β, where the value of β is known and the value of α is unknown $(\alpha > 0); T = \prod_{i=1}^{n} X_i$.

Solution.

The joint p.d.f is that same as that in Exercise 4. However, since the unknown parameter is now α instead of β, the appropriate factorization is now as follows:

$$f(\mathbf{x}|\alpha) = e^{-\beta\sum_{i=1}^{n} x_i}\frac{b^{n\alpha}}{(\Gamma(a))^n} T^{\alpha-1}$$

So the Factorization theorem works for

$$u(\mathbf{x}) = e^{-\beta\sum_{i=1}^{n} x_i} \text{ and } v(T,\alpha) = \frac{b^{n\alpha}}{(\Gamma(a))^n} T^{\alpha-1}$$

Exercise 6.

Consider a distribution for which the p.d.f. or the p.f. is $f(x \mid \theta)$, where the parameter θ is a k-dimensional vector belonging to some parameter space Ω. It is said that the family of distributions indexed by the values of θ in Ω is a k-parameter exponential family, or a k-parameter Koopman-Darmois family, if $f(x \mid \theta)$ can be written as follows for $\theta \in \Omega$ and all values of x :

$$f(x \mid \theta) = a(\theta)b(x)\exp\left[\sum_{i=1}^{k} c_i(\theta)d_i(x)\right].$$

Here, a and $c_1, \ldots, c_k$ are arbitrary functions of θ, and b and $d_1, \ldots, d_k$ are arbitrary functions of x. Suppose now that $X_1, \ldots, X_n$ form a random sample from a distribution which belongs to a k-parameter exponential family of this type, and define the k statistics $T_1, \ldots, T_k$ as follows:

$$T_i = \sum_{j=1}^{n} d_i\left(X_j\right) \quad \text{for } i = 1, \ldots, k.$$

Show that the statistics $T_1, \ldots, T_k$ are jointly sufficient statistics for θ.

Solution.

The joint p.d.f or p.f is

$$\prod_{j=1}^{n} b(x_j)[a(\theta)]^n exp\left[\sum_{i=1}^{k} c_i(\theta)\sum_{j=1}^{n} d_i(x_j)\right]$$

So it follows from the factorization criterion for

$$u(\mathbf{x}) = \prod_{j=1}^{n} b(x_j) \text{ and } v(T_1, ..., T_k, \theta) = [a(\theta)]^n exp\left[\sum_{i=1}^{k} T_i c_i(\theta)\right]$$

Exercise 7.

Show that each of the following families of distributions is a two-parameter exponential family as defined in Exercise 6:
(a) The family of all normal distributions for which both the mean and the variance are unknown
(b) The family of all gamma distributions for which both α and β are unknown
(c) The family of all beta distributions for which both α and β are unknown.

Solution.

In each part we shall present the p.d.f, and then we shall identify the functions a, b, c_1, c_2, d_1, d_2 in the form for a two-parameter exponential family given in Exercise 6.

(a) Let $\theta = (\mu, \sigma^2)$. Then

$$f(x|\theta) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Therefore

$$a(\theta) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left[-\frac{\mu^2}{2\sigma^2}\right]$$
$$b(x) = 1$$
$$c_1(\theta) = -\frac{1}{2\sigma^2}$$
$$d_1(x) = x^2$$
$$c_2(\theta) = \frac{\mu}{\sigma^2}$$
$$d_2(x) = x$$

(b) Let $\theta = (\alpha, \beta)$. Then

$$f(x|\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

Therefore

$$a(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}$$
$$b(x) = 1$$
$$c_1(\theta) = \alpha - 1$$
$$d_1(x) = logx$$
$$c_2(\theta) = -\beta$$
$$d_2(x) = x$$

(c) Let $\theta = (\alpha, \beta)$. Then

$$f(x|\theta) = \frac{1}{B(\alpha, \beta)} x^{a-1}(1-x)^{b-1}$$

Therefore

$$a(\theta) = \frac{1}{B(\alpha, \beta)}$$
$$b(x) = 1$$
$$c_1(\theta) = \alpha - 1$$
$$d_1(x) = logx$$
$$c_2(\theta) = \beta - 1$$
$$d_2(x) = log(1-x)$$