Parametric Statistics-Midterm Exam (Solutions)

Question 1.

The joint and marginal pmf's of X and Y are partly given in the following table:

Y \ X	1	2	3	
1	1/6	0		1/3
2		1/4		1/3
3			1/4	
	1/6	1/3		

(a) Complete the table.
(b) Are X and Y independent?
(c) Compute $E(X^2)$.
(d) Compute $E(X|Y=1)$.

Solution.

(a)

Y \ X	1	2	3	
1	1/6	0	**1/6**	1/3
2	**0**	1/4	**1/12**	1/3
3	**0**	**1/12**	1/4	**1/3**
	1/6	1/3	**1/2**	

(b) X and Y are independent if and only if for each x, y with $P(Y=y) > 0$ it stands that $P(X=x|Y=y) = P(X=x)$. Take $x=1, y=1$ and note that $P(X=1|Y=1) = \frac{P(X=1,Y=1)}{P(Y=1)} = \frac{1}{2}$ but $P(X=1) = \frac{1}{6}$. So X and Y are not independent.

(c) $E(X^2) = \sum_{i=1}^{3} x_i^2 P(X=x_i) = 6$.

(d) $E(X|Y=1) = \sum_{i=1}^{3} x_i P(X=x_i|Y=1) = 2$

Question 2.

Let $X_1, ..., X_n$ i.i.d with $X_i \sim Poisson(\lambda)$.
(a) Obtain a method of moments estimator for λ.
(b) Obtain a maximum likelihood estimator for λ.
(c) Is the maximum likelihood estimator unbiased?
(d) Assume that you are Bayesian and your prior for λ is a Gamma distribution with hyper-parameters a,b. Find the posterior distribution for λ.
(e) Find the Bayes estimator for the squared error loss.
(f) Is this Bayes estimator unbiased?.

Solution.

(a) We have 1 parameter and so 1 equation: $E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$. So $\lambda_{MOM} = \bar{X}_n$.

(b) We know that $f(x_i|\lambda) = \frac{1}{x_i!}\lambda^{x_i}e^{-\lambda}$.
The likelihood is

$$f(\mathbf{X}|\lambda) = \prod_{i=1}^{n} f(x_i|\lambda) = \frac{1}{\prod_{i=1}^{n} x_i!}e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}$$

and so

$$l(\lambda) = logf(\mathbf{X}|\lambda) = log\left[\frac{1}{\prod_{i=1}^{n} x_i!}\right] - n\lambda + log\lambda\sum_{i=1}^{n} x_i$$

And for the M.L.E we solve the equation:

$$l'(\lambda) = 0$$

$$-n + \frac{\sum_{i=1}^{n} x_i}{\lambda} = 0$$

and so $\lambda_{MLE} = \bar{X}_n$.

(c) For an estimator $\hat{\lambda}$ to be unbiased it must be $E(\hat{\lambda}) = \lambda$. It is obvious that $\hat{\lambda} = \lambda_{MLE}$ is an unbiased estimator because

$$E(\hat{\lambda}) = E(\bar{X}_n) = \frac{1}{n}E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}n\lambda = \lambda$$

(d) Prior: $\lambda \sim Gamma(a, b)$ and so $\pi(\lambda) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}$
Likelihood:

$$f(\mathbf{X}|\lambda) = \frac{1}{\prod_{i=1}^{n} x_i!}e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}$$

Posterior:

$$\pi(\lambda|\mathbf{X}) \propto \pi(\lambda)f(\mathbf{X}|\lambda) \propto e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}\lambda^{a-1}e^{-b\lambda} = \lambda^{\sum_{i=1}^{n} x_i + a - 1}e^{-(n+b)\lambda}$$

So $\lambda|\mathbf{X} \sim Gamma(\sum_{i=1}^{n} x_i + a, n + b)$.

(e) The Bayes estimator $\hat{\lambda}$ for S.E.L is the mean of posterior distribution. So

$$\hat{\lambda} = E(\lambda|\mathbf{X}) = \frac{\sum_{i=1}^{n} x_i + a}{n + b}$$

(f)

$$E(\hat{\lambda}) = E\left(\frac{\sum_{i=1}^{n} x_i + a}{n + b}\right) = \frac{1}{n + b}E\left(\sum_{i=1}^{n} x_i + a\right) = \frac{1}{n + b}\left(\sum_{i=1}^{n} E(x_i) + a\right) = \frac{n\lambda + a}{n + b}$$

So $\hat{\lambda}$ is not unbiased.

Question 3.

In Heraklion, 75 percent of people live in the city and 25 percent of people live in the suburbs. If there are 1200 random people from Heraklion at a particular concert, what is the probability that less than 270 people from the suburbs are attending the concert?

Solution.

Let X denote the number of people in the sample that are from the suburbs. Then $X \sim Bin(n, p)$ with $n = 1200$ and $p = 0.25$. Also

$$E(X) = np = 300 \text{ and } \sigma^2(X) = np(1-p) = 225$$

So by C.L.T,

$$Z = \frac{X - 300}{15}$$

is a standard normal and so

$$P(X \leq 270) = P(Z \leq -2) = 0.0227$$