# Parametric Statistics-Recitation 5 (Solutions)

## Exercise 1.

Suppose that $X$ is a random variable for which

$$\Pr(X \geq 0) = 1 \text{ and } \Pr(X \geq 10) = 1/5.$$

Prove that $E(X) \geq 2$.

## Solution

Since $\Pr(X \geq 0) = 1$ we can use the Markov inequality and so

$$E(X) \geq 10Pr(X \geq 10) = 2$$

## Exercise 2.

Suppose that $X_1, \ldots, X_n$ form a random sample of size $n$ from a distribution for which the mean is 6.5 and the variance is 4 . Determine how large the value of $n$ must be in order for the following relation to be satisfied:

$$\Pr\left(6 \leq \bar{X}_n \leq 7\right) \geq 0.8.$$

## Solution

Since $\mu = 6.5$ by Chebyshev inequality we have that

$$Pr(6 \leq \bar{X}_n \leq 7) = Pr\left(|\bar{X}_n - \mu| \leq \frac{1}{2}\right) \geq 1 - 4\sigma^2(\bar{X}_n) = 1 - \frac{16}{n}$$

Therefore we must have $1 - \frac{16}{n} \geq 0.8$ or $n \geq 80$.

## Exercise 3.

Suppose that 30 percent of the items in a large manufactured lot are of poor quality. Suppose also that a random sample of $n$ items is to be taken from the lot, and let $Q_n$ denote the proportion of the items in the sample that are of poor quality. Find a value of $n$ such that $\Pr(0.2 \leq Q_n \leq 0.4) \geq 0.75$ by using (a) the Chebyshev inequality and (b) the tables of the binomial distribution at the end of this book.

## Solution

(a) Here $E(Q_n) = 0.3$ and $Var(Q_n) = \frac{0.3 \cdot 0.7}{n} = \frac{0.21}{n}$. Therefore,

$$Pr(0.2 \leq Q_n \leq 0.4) = Pr(|Q_n - E(Q_n)| \leq 0.1) \geq 1 - \frac{0.21}{n \cdot 0.01} = 1 - \frac{21}{n}$$

and it must be $1 - \frac{21}{n} \geq 0.75$ or $n \geq 84$.

(b) Let $X_n$ denote the total number of items in the sample that are of poor quality. Then $X_n = nQ_n$ and

$$Pr(0.2 \leq Q_n \leq 0.4) = Pr(0.2n \leq X_n \leq 0.4n)$$

Since $X_n$ has a binomial distribution with parameters $n$ and $p = 0.3$, the value of this probability can be determined for various values of $n$ from the table of the binomial distribution. For $n = 15$ we found that

$$Pr(0.2n \leq X_n \leq 0.4n) = Pr(3 \leq X_n \leq 6) = 0.7419$$

and for $n = 20$

$$Pr(0.2n \leq X_n \leq 0.4n) = Pr(4 \leq X_n \leq 8) = 0.7796$$

Since this probability must be at least 0.75 we can take $n \geq 20$.

## Exercise 4.

It is said that a sequence of random variables $Z_1, Z_2, \ldots$ converges to a constant $b$ in quadratic mean if

$$\lim_{n\to\infty} E\left[(Z_n - b)^2\right] = 0.$$

Show that this is satisfied if and only if

$$\lim_{n\to\infty} E(Z_n) = b \text{ and } \lim_{n\to\infty} \operatorname{Var}(Z_n) = 0.$$

Hint: Use Exercise 5 of Sec. 4.3 in the DeGroot and Schervish book.

## Solution

Exercise 5 of Sec. 4.3 says that $E[(Z_n - b)^2] = [E(Z_n) - b]^2 + Var(Z_n)$. The proof is simple:

$$E[(Z_n - b)^2] = E[Z_n^2 - 2Z_n b + b^2] = E(Z_n^2) - 2bE(Z_n) + b^2 = Var(Z_n) + [E(Z_n)]^2 - 2bE(Z_n) + b^2 = [E(Z_n) - b]^2 + Var(Z_n)$$

Therefore $\lim_{n\to\infty} E\left[(Z_n - b)^2\right] = 0$ if and only if $\lim_{n\to\infty}[E(Z_n - b)]^2 = 0$ and $\lim_{n\to\infty} Var(Z_n) = 0$. That is

$$\lim_{n\to\infty} E(Z_n) = b \text{ and } \lim_{n\to\infty} \operatorname{Var}(Z_n) = 0.$$

## Exercise 5.

Suppose that the distribution of the number of defects on any given bolt of cloth is the Poisson distribution with mean 5 , and the number of defects on each bolt is counted for a random sample of 125 bolts. Determine the probability that the average number of defects per bolt in the sample will be less than 5.5 .

## Solution

Since the variance of a Poisson distribution is equal to the mean, the number of defects on any bolt has mean 5 and variance 5. Therefore, the distribution of the average number $\bar{X}_n$ on the 125 bolts will be approximately the normal distribution with mean 5 and variance $\frac{5}{125} = \frac{1}{25}$. If we let

$$Z = \frac{\bar{X}_n - 5}{\frac{1}{5}} = 5(\bar{X}_n - 5)$$

then $Z \sim N(0,1)$. Therefore

$$Pr(\bar{X}_n < 5.5) = Pr(Z < 2.5) \approx 0.9938$$

## Exercise 6.

A random sample of $n$ items is to be taken from a distribution with mean $\mu$ and standard deviation $\sigma$.

(a) Use the Chebyshev inequality to determine the smallest number of items $n$ that must be taken in order to satisfy the following relation:

$$\Pr\left(\left|\bar{X}_n - \mu\right| \leq \frac{\sigma}{4}\right) \geq 0.99.$$

(b) Use the central limit theorem to determine the smallest number of items $n$ that must be taken in order to satisfy the relation in part (a) approximately.

## Solution

(a) From Chebyshev inequality we have that

$$\Pr\left(\left|\bar{X}_n - \mu\right| > \frac{\sigma}{4}\right) \leq \frac{\sigma^2}{n} \cdot \left(\frac{4}{\sigma}\right)^2 = \frac{16}{n}$$

Therefore

$$\Pr\left(\left|\bar{X}_n - \mu\right| \leq \frac{\sigma}{4}\right) \geq 1 - \frac{16}{n}$$

So we want $1 - \frac{16}{n} \geq 0.99$ or $n \geq 1600$.

(b) We have that

$$\Pr\left(\left|\bar{X}_n - \mu\right| \leq \frac{\sigma}{4}\right) = Pr\left(|Z| \leq \frac{\sqrt{n}}{4}\right) \geq 0.99$$

where $Z \sim N(0,1)$. So from the table of standard normal distribution we found that $\frac{\sqrt{n}}{4} \geq 2.567$ or $n \geq 105.4$. Therefore, the smallest possible sample size is 106.

## Exercise 7.

Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with unknown mean $\theta$ and variance $\sigma^2$. Assuming that $\theta \neq 0$, determine the asymptotic distribution of $\bar{X}_n^3$.

## Solution

We are asking for the asymptotic distribution of $g(\bar{X}_n)$, where $g(x) = x^3$. The distribution of $\bar{X}_n$ is normal with mean $\theta$ and variance $\frac{\sigma^2}{n}$. According to the delta method, the asymptotic distribution of $g(\bar{X}_n)$ should be normal distribution with mean $g(\theta) = \theta^3$ and variance $\frac{\sigma^2}{n} \cdot [g'(\theta)]^2 = 9\theta^4 \cdot \frac{\sigma^2}{n}$.