Recitation 3

1. Suppose that X and Y have a continuous joint distribution for which the joint p.d.f. is defined as follows:

$$f(x,y) = \begin{cases} cy^2 & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine (a) the value of the constant c; (b) $\Pr(X+Y>2)$; (c) $\Pr(Y<1/2)$; (d) $\Pr(X \le 1)$; (e) $\Pr(X=3Y)$.

2. Suppose that the joint p.d.f. of X and Y is as follows:

$$f(x,y) = \begin{cases} 2xe^{-y} & \text{for } 0 \le x \le 1 \text{ and } 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

3. Suppose that the joint p.d.f. of X and Y is as follows:

$$f(x,y) = \begin{cases} 24xy & \text{for } x \ge 0, y \ge 0, \text{ and } x+y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

4. Each student in a certain high school was classified according to her year in school (freshman, sophomore, junior, or senior) and according to the number of times that she had visited a certain museum (never, once, or more than once). The proportions of students in the various classifications are given in the following table:

	Never	Once	More than once
Freshmen	0.08	0.10	0.04
Sophomores	0.04	0.10	0.04
Juniors	0.04	0.20	0.09
Seniors	0.02	0.15	0.10

(a) If a student selected at random from the high school is a junior, what is the probability that she has never visited the museum?

(b) If a student selected at random from the high school has visited the museum three times, what is the probability that she is a senior?

5. Let Y be the rate (calls per hour) at which calls arrive at a switchboard. Let X be the number of calls during a two-hour period. Suppose that the marginal p.d.f. of Y is

$$f_2(y) = \begin{cases} e^{-y} & \text{if } y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that the conditional p.f. of X given $Y = y$ is

$$g_1(x \mid y) = \begin{cases} \frac{(2y)^x}{x!} e^{-2y} & \text{if } x = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal p.f. of X. (You may use the formula $\int_0^\infty y^k e^{-y} dy = k!$!)

(b) Find the conditional p.d.f. $g_2(y \mid 0)$ of Y given $X = 0$.

(c) Find the conditional p.d.f. $g_2(y \mid 1)$ of Y given $X = 1$.

(d) For what values of y is $g_2(y \mid 1) > g_2(y \mid 0)$? Does this agree with the intuition that the more calls you see, the higher you should think the rate is?

6. Suppose that a random variable X can have each of the seven values $-3, -2, -1, 0, 1, 2, 3$ with equal probability. Determine the p.f. of $Y = X^2 - X$.

7. Suppose that the p.d.f. of X is as follows:

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

   Determine the p.d.f. of $Y = X^{1/2}$.

8. Let X and Y be random variables for which the joint p.d.f. is as follows:

$$f(x, y) = \begin{cases} 2(x + y) & \text{for } 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Find the p.d.f. of $Z = X + Y$.

9. Suppose that X_1 and X_2 are i.i.d. random variables and that the p.d.f. of each of them is as follows:

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Find the p.d.f. of $Y = X_1 - X_2$.

10. Suppose that a point is chosen at random on a stick of unit length and that the stick is broken into two pieces at that point. Find the expected value of the length of the longer piece.

11. Suppose that a particle is released at the origin of the xy-plane and travels into the half-plane where $x > 0$. Suppose that the particle travels in a straight line and that the angle between the positive half of the x-axis and this line is α, which can be either positive or negative. Suppose, finally, that the angle α has the uniform distribution on the interval $[-\pi/2, \pi/2]$. Let Y be the ordinate of the point at which the particle hits the vertical line $x = 1$. Show that the distribution of Y is a Cauchy distribution.

12. Suppose that a class contains 10 boys and 15 girls, and suppose that eight students are to be selected at random from the class without replacement. Let X denote the number of boys that are selected, and let Y denote the number of girls that are selected. Find $E(X - Y)$.

13. Suppose that one word is selected at random from the sentence THE GIRL. PUT ON HER BEAUTIFUL RED HAT. If X denotes the number of letters in the word that is selected, what is the value of $\text{Var}(X)$?

14. Suppose that X is a random variable for which $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$. Show that $E[X(X - 1)] = \mu(\mu - 1) + \sigma^2$.

15. Suppose that X and Y are independent random variables whose variances exist and such that $E(X) = E(Y)$. Show that

$$E\left[(X - Y)^2\right] = \text{Var}(X) + \text{Var}(Y).$$

16. For all random variables X and Y and all constants a, b, c, and d, show that

$$\text{Cov}(aX + b, cY + d) = ac\,\text{Cov}(X, Y).$$

17. Let X, Y, and Z be three random variables such that $\text{Cov}(X, Z)$ and $\text{Cov}(Y, Z)$ exist, and let a, b, and c be arbitrary given constants. Show that

$$\text{Cov}(aX + bY + c, Z) = a\,\text{Cov}(X, Z) + b\,\text{Cov}(Y, Z)$$