Parametric Statistics

Expectation and Variance

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Last time

- Two random variables have a bivariate joint distribution.
- More than two RVs have a multivariate joint distribution.
- We can compute marginal, conditional distributions from the joint pf.
- Independence is defined for RVs.
- Functions of RVs are RVs.

Lecture Summary

4.1 Expectations

4.2 Properties of Expectations

4.3 Variance

4.6 Covariance and Correlation

The Binomial Distribution

Example.

- Consider a machine that produces a defective item with probability $p (0 < p < 1)$ and a nondefective item with probability $1 - p$.
- Each item is independent of each other.
- Let 0 denote a non-defective item, and 1 denote a defective item.
- What is the probability of x defective items in n items?
- Example: $P(X = 2)$ if $n = 4$.

Binomial Distribution

We say that X is a binomial RV.

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Therefore, the p.f. of X will be as follows:

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Expectation

Expectation of a discrete RV

The **expected value** or **mean** or **first moment** of X is defined to be

$$E(X) = \sum_x x f(x)$$

assuming that the sum is well-defined.

- We can think of the expectation as the average of a very large number of independent draws from the distribution (i.i.d. draws).
- Example: $X \sim Bernoulli(p)$. $E(X)$?
- Example: Let X be the number of heads in 3 tosses. $E(X)$?

Expectation

Expectation of a continuous RV

The **expected value** or **mean** or **first moment** of X is defined to be

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

- Example: $X \sim Uniform[a, b]$. $E(X)$?

Expectation of a function of a random variable

Sometimes we are interested in the expectation of a function of a random variable $Y = r(X)$. One way to find the expectation of this random variable:

- Find its pmf $f(y)$
- Compute $\sum_y y f(y)$

Example

You play a game with three outcomes, $X \in \{3, 4, 6\}$ with

$$f(X = 3) = 0.4, f(X = 4) = 0.5, f(X = 6) = 0.1$$

Every time you play, you get $Y = X^2$. What are your expected gains?

Law Of The Unconscious Statistician

Let X be a random variable, and let r be a real-valued function of a real variable. If X has a continuous distribution, then

$$E(r(X)) = \int_{-\infty}^{\infty} r(x)f(x)dx,$$

if the mean exists. If X has a discrete distribution, then

$$E(r(X)) = \sum_{\text{All } x} r(x)f(x),$$

if the mean exists.

Law Of the Unconscious Statistician II

Suppose that $X_1, \ldots, X_n$ are random variables with the joint p.d.f. $f(x_1, \ldots, x_n)$. Let r be a real-valued function of n real variables, and suppose that $Y = r(X_1, \ldots, X_n)$. Then $E(Y)$ can be determined directly from the relation

$$E(Y) = \int \underset{R^n}{\cdots} \int r(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n,$$

if the mean exists. Similarly, if $X_1, \ldots, X_n$ have a discrete joint distribution with p.f. $f(x_1, \ldots, x_n)$, the mean of $Y = r(X_1, \ldots, X_n)$ is

$$E(Y) = \sum_{\text{All } x_1, \ldots, x_n} r(x_1, \ldots, x_n) f(x_1, \ldots, x_n),$$

if the mean exists.

Properties of Expectation

- $E(a) =$
- $E(aX) =$
- $E(aX + b) =$

Properties of Expectation

- $E(a) = a$
- $E(aX) = aE(X)$
- $E(aX + b) = aE(X) + b$
- $E(g(X)) \neq g(E(X))$ in most cases!
- **Jensen's Inequality.** Let g be a convex function, and let X be an RV with finite mean. Then $E[g(X)] \geq g(E(X))$.

Properties of Expectation

Linearity of Expectations

If $X_1, \ldots, X_n$ are n random variables such that each expectation $E(X_i)$ is finite $(i = 1, \ldots, n)$, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

Example: Find the expectation of the Binomial (N, p)

Expectation of a Product of Independent Random Variables

If $X_1, \ldots, X_n$ are n independent random variables such that each expectation $E(X_i)$ is finite $(i = 1, \ldots, n)$, then

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

Example

- If $X_1, X_2, \dots X_n$ are i.i.d. $Bernoulli(p)$ random variables then $Y = \sum_{i=1}^{n} X_i \sim Binomial(n, p)$

$$E(Y) = E(X_1) + \cdots + E(X_n) = np$$

Variance of a random variable

Sometimes we are also interested in quantifying how far from the mean

Definition (Variance/Standard Deviation)

Let X be a random variable with finite mean $\mu = E(X)$. The variance of X, denoted by $\text{Var}(X)$, is defined as follows:

$$\text{Var}(X) == E\left[(X - E(X))^2\right] = E\left[(X - \mu)^2\right].$$

If X has infinite mean or if the mean of X does not exist, we say that $\text{Var}(X)$ does not exist. The standard deviation of X is

$$\sigma_x = \sqrt{Var(X)}$$

if the variance exists.

- Let $X \sim Bernoulli(p)$. $Var(X) =?$
- Let X be the number of heads in three tosses. $Var(X) =?$

Alternative Method for Computing the Variance

For every random variable X, $\text{Var}(X) = E\left(X^2\right) - [E(X)]^2$.
Proof Let $E(X) = \mu$. Then

$$\begin{aligned} \text{Var}(X) &= E\left[(X-\mu)^2\right] \\ &= E\left(X^2 - 2\mu X + \mu^2\right) \\ &= E\left(X^2\right) - 2\mu E(X) + \mu^2 \\ &= E\left(X^2\right) - \mu^2. \end{aligned}$$

Properties of Variances

- $Var(a) =$
- $Var(aX + b) =$

Properties of Variances

- $Var(a) = 0$
- $Var(aX + b) = a^2 Var(X)$

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Linearity of variances for independent random variables.

Let $X_1, \ldots, X_n$ be a set of independent random variables. Then

$$Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n)$$

Let's prove it for the case of two discrete variables.

Covariance

Let X and Y be random variables having finite means. Let $E(X) = \mu_X$ and $E(Y) = \mu_Y$ The covariance of X and Y, which is denoted by $\mathrm{Cov}(X, Y)$, is defined as

$$\mathrm{Cov}(X, Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right],$$

if the expectation exists.

- For all random variables X and Y such that $\sigma_X^2 < \infty$ and $\sigma_Y^2 < \infty$,

  $$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

  .
- If X, Y are independent, then $Cov(X, Y) = 0$.

Correlation and Inequalities

Correlation

Covariance without dimensions

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

Schwartz inequality

$$[E(XY)]^2 \leq E(X^2)E(Y^2)]$$

Cauchy - Schwartz inequality

$$[Cov(X, Y)]^2 \leq \sigma_X^2 \sigma_Y^2$$

$$-1 \leq \rho(X, Y) \leq 1$$

Interquantile range

IQR

Let X be a random variable with quantile function $F^{-1}(p)$ for $0 < p < 1$. The interquartile range (IQR) is defined to be $F^{-1}(0.75) - F^{-1}(0.25)$. In words, the IQR is the length of the interval that contains the middle half of the distribution.

Recap

- Expectation is a summary of a distribution.
- We can compute the expectation of a function of an RV using LOTUS.
- Properties of expectation.
- Variance is a summary of how spread out a distribution is.
- Covariance describes how much two variables vary together.
- Correlation is covariance without scale.