3.3 The Cumulative Distribution Function

4. Suppose that the c.d.f. F of a random variable X is as sketched in Fig. 3.9. Find each of the following probabilities:

a. $\Pr(X = -1)$ b. $\Pr(X < 0)$
c. $\Pr(X \le 0)$ d. $\Pr(X = 1)$
e. $\Pr(0 < X \le 3)$ f. $\Pr(0 < X < 3)$
g. $\Pr(0 \le X \le 3)$ h. $\Pr(1 < X \le 2)$
i. $\Pr(1 \le X \le 2)$ j. $\Pr(X > 5)$
k. $\Pr(X \ge 5)$ l. $\Pr(3 \le X \le 4)$

6. Suppose that the c.d.f. of a random variable X is as follows:

$$F(x) = \begin{cases} e^{x-3} & \text{for } x \le 3, \\ 1 & \text{for } x > 3. \end{cases}$$

Find and sketch the p.d.f. of X.

15. Suppose that X has the p.d.f.

$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find and sketch the c.d.f. or X.

4.1 The Expectation of a Random Variable

4. Suppose that one word is to be selected at random from the sentence THE GIRL PUT ON HER BEAUTIFUL RED HAT. If X denotes the number of letters in the word that is selected, what is the value of $E(X)$?

6. Suppose that a random variable X has a continuous distribution with the p.d.f. f given in Example 4.1.6. Find the expectation of $1/X$.

Example 4.1.6 Expected Failure Time. An appliance has a maximum lifetime of one year. The time X until it fails is a random variable with a continuous distribution having p.d.f.

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E(X) = \int_0^1 x(2x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}.$$

We can also say that the expectation of the distribution with p.d.f. f is 2/3. ◄

4.2 Properties of Expectations

3. Suppose that three random variables X_1, X_2, X_3 form a random sample from the uniform distribution on the interval [0, 1]. Determine the value of

$E[(X_1 - 2X_2 + X_3)^2]$.

8. Suppose that a class contains 10 boys and 15 girls, and suppose that eight students are to be selected at random from the class without replacement. Let X denote the number of boys that are selected, and let Y denote the number of girls that are selected. Find $E(X - Y)$.

4.3 Variance

2. Suppose that one word is selected at random from the sentence THE GIRL PUT ON HER BEAUTIFUL RED HAT. If X denotes the number of letters in the word that is selected, what is the value of $\mathrm{Var}(X)$?

4. Suppose that X is a random variable for which $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Show that $E[X(X-1)] = \mu(\mu - 1) + \sigma^2$.

6. Suppose that X and Y are independent random variables whose variances exist and such that $E(X) = E(Y)$. Show that

$$E[(X - Y)^2] = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

4.4 Moments

8. Suppose that X is a random variable for which the m.g.f. is as follows:

$$\psi(t) = e^{t^2+3t} \quad \text{for } -\infty < t < \infty.$$

Find the mean and the variance of X.

10. Suppose that the random variables X and Y are i.i.d. and that the m.g.f. of each is

$$\psi(t) = e^{t^2+3t} \quad \text{for } -\infty < t < \infty.$$

Find the m.g.f. of $Z = 2X - 3Y + 4$.

4.5 The Mean and the Median

2. Suppose that a random variable X has a discrete distribution for which the p.f. is as follows:

$$f(x) = \begin{cases} cx & \text{for } x = 1, 2, 3, 4, 5, 6, \\ 0 & \text{otherwise.} \end{cases}$$

Determine all the medians of this distribution.

6. Suppose that a random variable X has a continuous distribution for which the p.d.f. f is as follows:

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of d that minimizes **(a)** $E[(X-d)^2]$ and **(b)** $E(|X-d|)$.

4.6 Covariance and Correlation

5. For all random variables X and Y and all constants a, b, c, and d, show that

$$\mathrm{Cov}(aX + b, cY + d) = ac\,\mathrm{Cov}(X, Y).$$

12. Suppose that X and Y have a continuous joint distribution for which the joint p.d.f. is as follows:

$$f(x, y) = \begin{cases} \frac{1}{3}(x + y) & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of $\mathrm{Var}(2X - 3Y + 8)$.

7. Let X, Y, and Z be three random variables such that $\mathrm{Cov}(X, Z)$ and $\mathrm{Cov}(Y, Z)$ exist, and let a, b, and c be arbitrary given constants. Show that

$$\mathrm{Cov}(aX + bY + c, Z) = a\,\mathrm{Cov}(X, Z) + b\,\mathrm{Cov}(Y, Z).$$

18. Let X and Y have a continuous distribution with joint p.d.f.

$$f(x, y) = \begin{cases} x + y & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the covariance $\mathrm{Cov}(X, Y)$.

4.7 Conditional Expectation

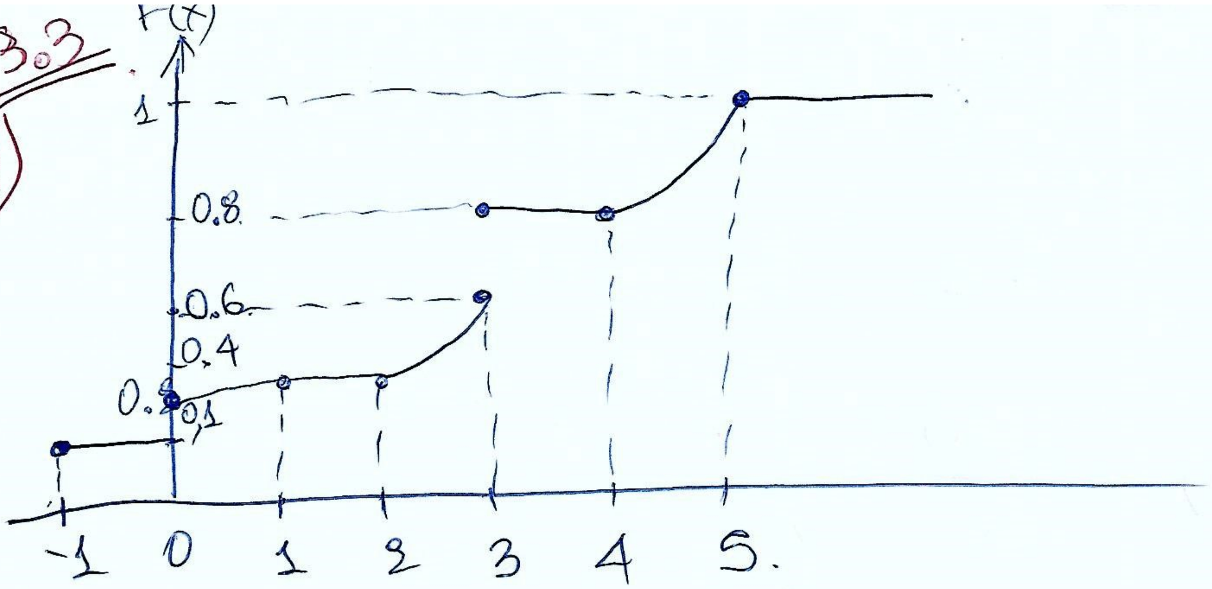
7. Suppose that X and Y have a continuous joint distribution for which the joint p.d.f. is as follows:

$$f(x, y) = \begin{cases} x + y & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(Y|X)$ and $\mathrm{Var}(Y|X)$.

3.3

4)



- $P_r(X \leq x) = F(x)$
- $P_r(x_1 \leq X \leq x_2) = F(x_2) - F(x_1)$.
- $P_r(X < x) = F(x)$.
- $P_r(X = x) = F(x) - F(x^-)$.
- $\frac{dF(x)}{dx} = f(x)$.

a) $P_r(X=-1) = F(-1) - F(-1^-) = 0,1$.

b) $P_r(X<0) = F(0^-) = 0,1$.

c) $P_r(X \leq 0) = F(0) = 0,2$

d) $P_r(x=1) = 0$

e) $P_r(0 < x \leq 3) = F(3) - F(0) = 0,8 - 0.2$

j) $P_r(X>5) = 1 - F(5) = 1 - 1 = 0$.

4.5

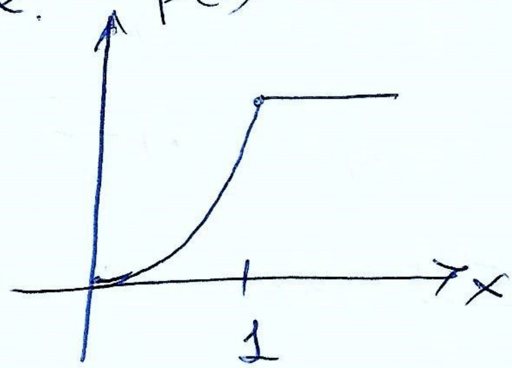
6) $F(x) = \begin{cases} e^{x-3} & \text{for } x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} e^{x-3}, & x < 3 \\ 0, & x > 3 \end{cases}$$

4.5

15) $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$$F(x) = \int_0^x 2y\,dy = x^2.$$



4.1

4) The expectation of X : $E(x) = \int_{-\infty}^{\infty} x f(x) dx$

THE GIRL PUT ON HER BEAUTIFUL RED HAT.

X	f(x)
2	1/8
3	5/8
4	1/8
9	1/8

$$E(x) = 2 \cdot \left(\frac{1}{8}\right) + 3 \cdot \left(\frac{5}{8}\right) + 4\left(\frac{1}{8}\right) + 9 \cdot \left(\frac{1}{8}\right) =$$

$$= 3{,}75$$

4.5

8) 10 boys
15 girls
select random 8.

8 random variable $X_1, X_2, \dots X_8$.

$X_1 = 1$ if boy selected

$$P = \frac{10}{25}.$$

$$P_r(X_i = 1) = \frac{10}{25}.$$

$$P_r(X_1 = 0) = \frac{15}{25}$$

$$X = X_1 + X_2 + \dots + X_8$$

$$E(X) = E(X_1) + \dots + E(X_8) = 8\left(\frac{10}{25}\right) = \frac{16}{5}$$

$$Y = 8 - X.$$

$$E[Y] = 8 - E[X] = 8 - \frac{16}{5}$$

4.3

2) Var(X).

$$Var(X) = E(X^2) - [E(X)]^2.$$

$$Var(X) = E[(X-\mu)^2].$$

$$E(X^2) = 2^2\left(\frac{1}{8}\right) + 3^2\left(\frac{5}{8}\right) + 4^2\left(\frac{1}{8}\right) + 9^2\left(\frac{1}{8}\right) = \frac{73}{4}.$$

$$E(X) = \frac{15}{4}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{73}{4} - \left(\frac{15}{4}\right)^2 = \frac{67}{16}$$

4.6

- $cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$.
- $cov(X,Y) = E(XY) - E(X)E(Y)$.
- $\rho(X,Y) = \dfrac{cov(X,Y)}{\sigma_x \cdot \sigma_y}$

$|\rho(X,Y)| = 1$ ~ correlated.

12) $f(x,y) = \begin{cases} \frac{1}{3}(x+y) & : \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 2. \\ 0 & , \text{otherwise.} \end{cases}$

$Var(2X-3Y+8) = 4Var(X) + 9Var(Y) - 2\cdot 2\cdot 3 Cov(X,Y)$

know: $Var(X) = E[(X-\mu)^2]$.

$Var(aX+bY+c) = E[\{(aX+bY+c)-(a\mu_x+b\mu_y+c)\}^2]$

$= E[\{a(X-\mu_x)+b(Y-\mu_y)\}^2] =$

$= E[a^2(X-\mu_x)^2 + b^2(Y-\mu_y)^2 + 2ab(X-\mu_x)(Y-\mu_y)] =$

$= a^2 Var(X) + b^2 Var(Y) + 2ab Cov(X,Y)$

$$E(X) = \int_0^1 \int_0^2 x \frac{1}{3}(x+y)\,dy\,dx = \frac{5}{9}$$

$$E(Y) = \int_0^1 \int_0^2 y \cdot \frac{1}{3}(x+y)\,dy\,dx = \frac{11}{9}$$

$$E(X^2) = \int_0^1 \int_0^2 x^2 \frac{1}{3}(x+y)\,dy\,dx = \frac{7}{18}$$

$$E(Y^2) = \int_0^1 \int_0^2 y^2 \frac{1}{3}(x+y)\,dy\,dx = \frac{16}{9}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_0^1 \int_0^2 xy \frac{1}{3}(x+y)\,dy\,dx = \frac{2}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}$$

$$Var(Y) = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{9}\right) = -\frac{1}{81}$$

$$E(Y|X) = \int_0^1 y\,g(y|x)\,dy = \int_0^1 \frac{2(xy+y^2)}{2x+1}\,dy =$$

$$= \frac{3x+2}{3(2x-1)}.$$

$$E(Y^2|X) = \int_0^1 \frac{2(xy^2+y^3)}{2x+1}\,dy = \frac{4x+3}{6(2x+1)}.$$

$$Var(Y|X) = E(Y^2|X) - \left[E(Y|X)\right]^2 =$$

$$= \frac{4x+3}{6(2x+1)} - \left(\frac{3x+2}{3(2x+1)}\right)^2 = \frac{1}{36}\left(3 - \frac{1}{(2x+1)^2}\right).$$

4.6

5) $Cov(aX+b, cY+d) = a\cdot c\, Cov(X,Y)$.

Απόδ.:

$$Cov(aX+b, cY+d) = E\left[(aX+b-a\mu_x-b)(cY+d-c\mu_y-d)\right] =$$

$$= E\left[ac(X-\mu_x)(Y-\mu_y)\right] = ac\, Cov(X,Y).$$

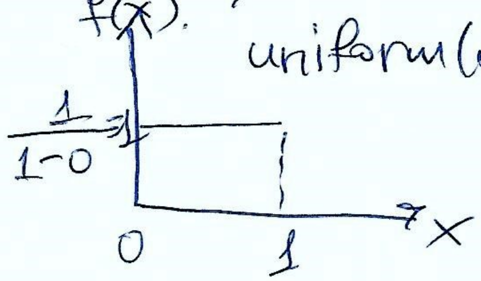
όπου $E(aX+b) = a\mu_x + b$.

$E(cY+d) = c\mu_y + d$.

4.2

3). X_1, X_2, X_3 random variables from uniform distribution on $[0,1]$.

$$E\left[(X_1-2X_2+X_3)^2\right] =$$



$$= E(X_1^2+4X_2^2+X_3^2-4X_1X_2 + 2X_1X_3-4X_2X_3) =$$

$$= E(X_1^2)+4E(X_2^2)+E(X_3^2)-4E(X_1X_2)+ + 2E(X_1X_3)-4E(X_2X_3).$$

X_1, X_2, X_3 independent : $E(X_i \cdot X_j) = E(X_i)\cdot E(X_j)$

$$E(X_1) = \int_0^1 x\,dx = \frac{1}{2}.$$

$$E(X_i^2) = \int_0^1 x^2\,dx = \frac{1}{3}$$

2.3

6) • machine is adjusted properly:

- 50% items high
- 50% items medium

• machine is adjusted improperly:

10% time :
- 25% high
- 75% medium

a) 5 items $\begin{cases} 4 \text{ high} \\ 1 \text{ medium} \end{cases}$

The probability was adjusted properly this time.

A_1 : denote the event that the machine is adjusted properly

A_2 : denote the event that the machine is adjusted improperly.

B : be the event that four of five is inspected items are of high quality.

$$Pr(A_1|B) = \frac{Pr(A_1)Pr(B|A_1)}{Pr(A_1)Pr(B|A_1) + Pr(A_2)Pr(B|A_2)} =$$

$$= \frac{0.9\binom{5}{4}(0.5)^5}{(0.9)\binom{5}{4}(0.5)^5 + 0.1\binom{5}{4}(0.25)^4(0.75)} = \frac{96}{97}$$

b) $Pr(A_1) = \frac{96}{97}$, $Pr(A_2) = \frac{1}{97}$

C : denote the event that the additional item of medium quality.

$$Pr(A_1|C) = \frac{Pr(A_1)\, Pr(C|A_1)}{Pr(A_1)\, Pr(C|A_1) + Pr(A_2)\, Pr(C|A_2)} =$$

$$= \frac{\frac{96}{97} \cdot \frac{1}{2}}{\frac{96}{97} \cdot \frac{1}{2} + \frac{1}{97} \cdot \frac{3}{4}} = \frac{64}{65}.$$