Parametric Statistics

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1.6 Finite Sample Spaces

1.7 Counting Methods

1.8 Combinatorial Methods

- SKIP: Multinomial Coefficients
- SKIP: 1.10 Probability of a Union of Events
- SKIP:1.11 Statistical Swindles

2.1 Conditional Probability

2.3 Independent Events

2.4 Bayes Rule

- SKIP: 2.4 The Gambler's Ruin Problem

Probability Measure

We want to assign a real number $\mathbb{P}(A)$ to every element A of a $\sigma-$algebra $\mathcal{A}$ to $[0,1]$ which represents how likely event A is to occur. This is called the probability of A.

- Axiom 1: $\mathbb{P}(A) \geq 0$ for every A
- Axiom 2: $\mathbb{P}(\Omega) = 1$
- Axiom 3: for an infinite sequence $A_1, A_2, \ldots$ of disjoint events

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

The triplet $(\Omega, \mathcal{A}, \mathbb{P})$ is called a **probability space**.

Discrete Probability Spaces.

For uncountable sample spaces, we need σ-algebras (which do not include all subsets of Ω) to avoid mathematical difficulties. Finite and countable sample spaces are much easier to think about:

Examples

- Consider a single toss of a coin. If we believe that heads (H) and tails (T) are equally likely, find an appropriate probability model.
- Consider a single roll of a die. if we believe that all six outcomes are equally likely, find an appropriate probability model.
- We toss an unbiased coin n times. What is an appropriate probability model?

Counting in a uniform (simple) probability space

In discrete sample spaces, if all outcomes of an experiments are considered equally likely, then for each event A,

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

Example:

We toss a (fair) coin twice. Ω has 36 elements: $11, 12, \ldots, 16, 21, \ldots, 26, \ldots, 61, \ldots, 66$

Let's say we are interested in the event "at least one 6" To assign a probability to each possible event, we need to be able to count: The number of points in Ω and A. To do so, we need some combinatorial methods.

Multiplication rule

General counting rule:

- r steps
- n_r choices at each step
- Then the number of choices are $n_1 \times n_2 \times \cdots \times n_r$

Example

- Number of license plates with 3 letters and 4 digits.
  - Choose from A,B, E, Z, H, I, K, M, N, O, P, T, Y, X.
- Without repetitions:
- **Permutations** of a set with n elements:
- Number of subsets of set $\{1, \ldots, n\}$:

Combinations

Combinations

$\binom{n}{k}$: Number of distinct ways of choosing k elements from a collection of n objects (binomial coefficient)

- Choose the k items one at a time:
- Choose k items, then order them ($k!$ possible orders):

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- $\sum_{k=0}^{n} \binom{n}{k}$:

Conditional Probability

You bought a lottery ticket with numbers $15, 1, 30, 26, 17$. You win the lottery if you get all numbers correctly. The numbers are 1-30 (no repetitions).

- What are your chances of winning?
- You see that the first number is 15, and then there is a power outage. What are your chances of winning now?
- We just described the **conditional probability** $P(A|B)$

Conditional Probability

Definition (Conditional Probability of A given B)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The probability of event A in the universe (sample space) where event B has already happened.

Conditional Probabilities are behave like probabilities!

- $P(A|B) + P(A^c|B) = 1$
- $P(A \cup C|B) = P(A|B) + P(C|B) - P(A \cap C|B)$
- $P(B|B) = 1$

Multiplication Rule for Conditional Probabilities

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A)P(B|A)$$

$$P(A \cap B) = P(B)P(A|B)$$

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

$$P(\bigcap_{i=1}^{n} A_i) = \prod_{k=1}^{n} P(A_i|\bigcap_{j=1}^{k-1} A_j)$$

Multiplication Rule Example

Example

- You toss two coins. What is the probability that the second toss is "heads" given that the first toss is heads?
- Coin tosses are independent (by construction).

Definition

Two events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

We denote this as

$$A \perp\!\!\!\perp B$$

A set of events $\{A_i : i \in I\}$ is independent if

$$\mathbb{P}(\cap_{i \in J} A_i) = \prod \mathbb{P}(A_i)$$

for every finite subset J of I.

Independence as conditional probability

We already discussed how if we flip a coin twice, the probability of two heads is $\frac{1}{2} \times \frac{1}{2}$. This is because we consider the two tosses as independent. This means, the outcome of the first coin flip does not affect the outcome of the second flip.

- P(Second toss is heads|first toss is heads) = 1/2 = P(Second toss is heads)
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$
- If I tell you that B has happened, this does not change your belief about how likely A is.

Conditional Independence

Just Independence in the new universe

$$P(A \cap B|C) = P(A|C)P(B|C)$$

Checking independence

- If we flip a coin twice, we typically assume that the flips are independent (i.e., the coin has no memory of previous flip).
- Sometimes, independence just comes up. Example: We roll a fair die and we are interested in the following two events: A : "The outcome is an even number" B : "The outcome is one of the numbers $\{1, 2, 3, 4\}$"
- How about disjoint events?

Multiplication Rule Example #2

Example

- You go past a clinic that gives tests for a rare (1 in 1000 people) disease. The clinic tells you that the tests are 95% accurate, so
  - $P(T^{+}|D+) = 0.95.$
  - $P(T^{-}|D^{-}) = 0.95$
- What is the probability that you test positive and you don't have the disease?
- Your test comes out positive. What is the probability that you have the disease?

Bayes Rule

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$

Let's prove it

Why is Bayes Rule so important?

Very often, people confuse $P(A|B)$ and $P(B|A)$. These can be VERY different.

Think about it:

You read in the paper: "Half of the people hospitalized with covid-19 are fully vaccinated". Do you think that getting the vaccine lowers your chances of getting hospitalized?

Why is Bayes Rule so important?

- Vacc: Yes if vaccinated, zero otherwise
- Hosp: Yes if hospitalized, zero otherwise.
- $P(Hosp|Vacc) = 0.01$
- $P(Hosp|\neg Vacc) = 0.2$
- Three different possibilities: $P(Vacc) = 0.8, 0.5, 0.99$

Let' s use Bayes rule to compute $P(Vacc|Hosp)$ for all three cases.

Review(1)

- Probability is a way to quantify the probability with which an event occurs.
- For discrete sample spaces, it is pretty easy to define a probability measure over the set of all possible events.
- We can use the axioms of probability to prove several properties of probability.

Review(2)

- Two events are called independent when knowing the value of one doesn't influence the probability of the value of the other.
- The conditional probability of A given B denotes the probability of event A in a world where B has occurred.
- Bayes rule connects $P(A|B)$ and $P(B|A)$. These two are confused but they are not the same.

Recitation Exercises

Section	Exercises
1.6	1, 6
1.7	4, 5, 6
1.8	2, 3, 4, 12
2.1	4, 14
2.2	2, 12
2.3	5, 6