Parametric Statistics-Recitation 9 (Solutions)

Exercise 1.

Suppose that a random sample is to be taken from the normal distribution with unknown mean θ and standard deviation 2. How large a random sample must be taken in order that $E_\theta\left(\left|\bar{X}_n-\theta\right|^2\right) \leq 0.1$ for every possible value of θ ?

Solution.

It is known that $\bar{X}_n$ has the normal distribution with mean θ and variance $\frac{4}{n}$. Therefore,

$$E_\theta\left(\left|\bar{X}_n-\theta\right|^2\right) = Var_\theta\left(\bar{X}_n\right) = \frac{4}{n}$$

So $\frac{4}{n} \leq 0.1$ if and only if $n \geq 40$.

Exercise 2.

Suppose that a random sample is to be taken from the Bernoulli distribution with unknown parameter p. Suppose also that it is believed that the value of p is in the neighborhood of 0.2 . How large a random sample must be taken in order that $\Pr\left(\left|\bar{X}_n-p\right| \leq 0.1\right) \geq 0.75$ when $p=0.2$?

Solution.

When $p=0.2$, the random variable $Z_n = n\bar{X}_n$ will have a binomial distribution with parameters n and $p=0.2$ and

$$\Pr\left(\left|\bar{X}_n-p\right| \leq 0.1\right) = Pr(0.1n \leq Z_n \leq 0.3n)$$

The value of n for which this probability will be at least 0.75 must be determined by trial and error from the table of the binomial distribution. For $n=8$ the probability becomes

$$Pr(0.8 \leq Z_8 \leq 2.4) = Pr(Z_8=1)+Pr(Z_8=2) = 0.3355+0.2936 = 0.6291$$

For $n=9$ we have

$$Pr(0.9 \leq Z_9 \leq 2.7) = Pr(Z_9=1)+Pr(Z_9=2) = 0.3020+0.3020 = 0.6040$$

For $n=10$ we have

$$Pr(1 \leq Z_{10} \leq 3) = Pr(Z_{10}=1)+Pr(Z_{10}=2)+Pr(Z_{10}=3) = 0.2684+0.3020+0.2013 = 0.7717$$

Hence $n=10$ is sufficient. It should be noted that although a sample size of $n=10$ will meet the required conditions, a sample size of $n=11$ will not meet the conditions because for $n=11$ we have

$$Pr(1.1 \leq Z_{11} \leq 3.3) = Pr(Z_{11}=2)+Pr(Z_{11}=3)$$

Thus, only 2 terms of the binomial distribution for $n=11$ are included, whereas 3 terms of binomial distribution for $n=10$ were included.

Exercise 3.

Suppose that a point (X, Y) is to be chosen at random in the xy-plane, where X and Y are independent random variables and each has the standard normal distribution. If a circle is drawn in the xy-plane with its center at the origin, what is the radius of the smallest circle that can be chosen in order for there to be probability 0.99 that the point (X, Y) will lie inside the circle?

Solution.

Let r denote the radius of the circle. The point (X, Y) will lie inside the circle if and only if $X^2 + Y^2 < r^2$. Also $X^2 + Y^2$ has a χ^2 distribution with 2 degrees of freedom. It is found from the table that

$$Pr(X^2 + Y^2 \leq 9.210) = 0.99$$

Therefore, we must have $r^2 = 9.210$.

Exercise 4.

Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2. Assuming that the sample size n is 16 , determine the values of the following probabilities:
a. $\Pr\left[\frac{1}{2}\sigma^2 \leq \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 \leq 2\sigma^2\right]$
b. $\Pr\left[\frac{1}{2}\sigma^2 \leq \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 \leq 2\sigma^2\right]$

Solution.

(a) Since $\frac{X_i - \mu}{\sigma}$ has a standard normal distribution for i=1,...,n then

$$W = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2}$$

has the χ^2 distribution with n degrees of freedom. The required probability can be rewritten as follows:

$$Pr\left(\frac{n}{2} \leq W \leq 2n\right)$$

Thus, then $n = 16$, we must evaluate $Pr(8 \leq W \leq 32) = Pr(W \leq 32) - Pr(W \leq 8)$ where W has the χ^2 distribution with 16 degrees of freedom. From the table we find that

$$Pr(8 \leq W \leq 32) = Pr(W \leq 32) - Pr(W \leq 8) = 0.99 - 0.05 = 0.985$$

(b) By theorem

$$V = \sum_{i=1}^{n} \frac{(X_i - \bar{X}_n)^2}{\sigma^2}$$

has the χ^2 distribution with n-1 degrees of freedom. The required probability can be rewritten as follows:

$$Pr\left(\frac{n}{2} \leq V \leq 2n\right)$$

Thus, then $n = 16$, we must evaluate $Pr(8 \leq V \leq 32) = Pr(V \leq 32) - Pr(V \leq 8)$ where V has the χ^2 distribution with 15 degrees of freedom. From the table we find that

$$Pr(8 \leq V \leq 32) = Pr(V \leq 32) - Pr(V \leq 8) = 0.993 - 0.079 = 0.914$$

Exercise 5.

Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2, and let $\widehat{\sigma^2}$ denote the sample variance. Determine the smallest values of n for which the following relations are satisfied: a. $\Pr\left(\frac{\widehat{\sigma^2}}{\sigma^2} \leq 1.5\right) \geq 0.95$ b. $\Pr\left(\left|\widehat{\sigma^2} - \sigma^2\right| \leq \frac{1}{2}\sigma^2\right) \geq 0.8$

Solution.

(a) The random variable $V = \frac{n\widehat{\sigma^2}}{\sigma^2}$ has a χ^2 distribution with n-1 degrees of freedom. The required probability can be written in the form

$$Pr(V \leq 1.5n) \geq 0.95$$

. By trial and error, it is found that for $n = 20$, V has 19 degrees of freedom and $Pr(V \leq 30) < 0.95$. However, for $n = 21$, V has 20 degrees of freedom and $Pr(V \leq 31.5) > 0.95$. Thus the smallest n is $n = 21$.

(b) The required probability can be written in the form

$$Pr\left(\frac{n}{2} \leq V \leq \frac{3n}{2}\right) = Pr\left(V \leq \frac{3n}{2}\right) - Pr\left(V \leq \frac{n}{2}\right)$$

where V again has a χ^2 distribution with n-1 degrees of freedom. By trial and error, it is found that for $n = 12$, V has 11 degrees of freedom and

$$Pr(V \leq 18) - Pr(V \leq 6) = 0.915 - 0.130 < 0.8$$

However, for $n = 13$, V has 12 degrees of freedom and

$$Pr(V \leq 19.5) - Pr(V \leq 6.5) = 0.919 - 0.113 > 0.8$$

Thus the smallest n is $n = 13$.

Exercise 6.

Suppose that a random sample of eight observations is taken from the normal distribution with unknown mean μ and unknown variance σ^2, and that the observed values are $3.1, 3.5, 2.6, 3.4, 3.8, 3.0, 2.9,$ and 2.2. Find the shortest confidence interval for μ with each of the following three confidence coefficients: (a) 0.90 , (b) 0.95 , and (c) 0.99 .

Solution.

In this exercise

$$\bar{X}_n = 3.0625 \text{ , } \sigma' = \left[\sum_{i=1}^{n} \frac{(X_i - \bar{X}_n)^2}{n-1}\right]^{1/2} = 0.5125 \text{ and } \frac{\sigma'}{n^{1/2}} = 0.1812$$

Therefore, the shortest confidence interval for μ will have the form

$$3.0625 - 0.1812c < \mu < 3.0625 + 0.1812c$$

If a confidence coefficient is γ then c satisfy the relation $Pr(-c < U < c) = \gamma$, where U has the t distribution with $n - 1$ degrees of freedom. It's equivalent to say that c satisfy

$$Pr(U \leq c) = 1 - \frac{1-\gamma}{2}$$

We have $n - 1 = 7$ degrees of freedom.

(a) Here $\gamma = 0.9$ and from $Pr(U \leq c) = 0.95$ it follows that $c = 1.895$. Therefore, the confidence interval for μ has endpoints

2.719 and 3.406

(b) Here $\gamma = 0.95$ and from $Pr(U \leq c) = 0.975$ it follows that $c = 2.365$. Therefore, the confidence interval for μ has endpoints

2.634 and 3.491

(c) Here $\gamma = 0.99$ and from $Pr(U \leq c) = 0.995$ it follows that $c = 3.499$. Therefore, the confidence interval for μ has endpoints

2.428 and 3.697

Exercise 7.

Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and unknown variance σ^2, and let the random variable L denote the length of the shortest confidence interval for μ that can be constructed from the observed values in the sample. Find the value of $E\left(L^2\right)$ for the following values of the sample size n and the confidence coefficient γ :
a. $n = 5, \gamma = 0.95$
b. $n = 10, \gamma = 0.95$
c. $n = 30, \gamma = 0.95$
d. $n = 8, \gamma = 0.90$
e. $n = 8, \gamma = 0.95$
f. $n = 8, \gamma = 0.99$

Solution.

The endpoints of the confidence interval are $\bar{X}_n - c\frac{\sigma'}{n^{1/2}}$ and $\bar{X}_n + c\frac{\sigma'}{n^{1/2}}$. Therefore

$$L = \frac{2c\sigma'}{n^{1/2}} \text{ and } L^2 = \frac{4c^2\sigma'^2}{n}$$

Since

$$W = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2}$$

has the χ^2 distribution with $n-1$ degrees of freedom, $E(W) = n - 1$. Therefore

$$E(\sigma') = E\left(\frac{\sigma^2 W}{n-1}\right) = \sigma^2$$

It follows that

$$E(L^2) = \frac{4c^2\sigma^2}{n}$$

(a) Here $c = 2.776$ and $E(L^2) = 6.16\sigma^2$.
(b) Here $c = 2.262$ and $E(L^2) = 2.05\sigma^2$.
(c) Here $c = 2.045$ and $E(L^2) = 0.56\sigma^2$.
(d) Here $c = 1.895$ and $E(L^2) = 1.80\sigma^2$.
(e) Here $c = 2.365$ and $E(L^2) = 2.80\sigma^2$.
(f) Here $c = 3.499$ and $E(L^2) = 6.12\sigma^2$.

Exercise 8.

Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and known variance σ^2. How large a random sample must be taken in order that there will be a confidence interval for μ with confidence coefficient 0.95 and length less than 0.01σ ?

Solution.

The length of the confidence interval is $L = \frac{2c\sigma}{\sqrt{n}}$. For $\gamma = 0.95$ it is known that c satisfies

$$Pr(Z \leq c) = 1 - \frac{1-\gamma}{2} = 0.975$$

So $c = 1.96$ and $L = \frac{3.92\sigma}{\sqrt{n}}$. Therefore

$$L < 0.01\sigma$$

if and only if

$$\sqrt{n} > 392 \text{ or } n > 153664$$