Recitation 9

1. Suppose that a random sample is to be taken from the normal distribution with unknown mean θ and standard deviation 2. How large a random sample must be taken in order that $E_\theta\left(\left|\bar{X}_n-\theta\right|^2\right) \leq 0.1$ for every possible value of θ ?

2. Suppose that a random sample is to be taken from the Bernoulli distribution with unknown parameter p. Suppose also that it is believed that the value of p is in the neighborhood of 0.2 . How large a random sample must be taken in order that $\operatorname{Pr}\left(\left|\bar{X}_n-p\right| \leq 0.1\right) \geq 0.75$ when $p=0.2$?

3. Suppose that a point (X, Y) is to be chosen at random in the xy-plane, where X and Y are independent random variables and each has the standard normal distribution. If a circle is drawn in the xy-plane with its center at the origin, what is the radius of the smallest circle that can be chosen in order for there to be probability 0.99 that the point (X, Y) will lie inside the circle?

4. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2. Assuming that the sample size n is 16 , determine the values of the following probabilities:
   a. $\operatorname{Pr}\left[\frac{1}{2} \sigma^2 \leq \frac{1}{n} \sum_{i=1}^n\left(X_i-\mu\right)^2 \leq 2 \sigma^2\right]$
   b. $\operatorname{Pr}\left[\frac{1}{2} \sigma^2 \leq \frac{1}{n} \sum_{i=1}^n\left(X_i-\bar{X}_n\right)^2 \leq 2 \sigma^2\right]$

5. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2, and let $\widehat{\sigma^2}$ denote the sample variance. Determine the smallest values of n for which the following relations are satisfied: a. $\operatorname{Pr}\left(\frac{\widehat{\sigma^2}}{\sigma^2} \leq 1.5\right) \geq 0.95$ b. $\operatorname{Pr}\left(\left|\widehat{\sigma^2}-\sigma^2\right| \leq \frac{1}{2} \sigma^2\right) \geq 0.8$

6. Suppose that a random sample of eight observations is taken from the normal distribution with unknown mean μ and unknown variance σ^2, and that the observed values are $3.1, 3.5, 2.6, 3.4, 3.8, 3.0, 2.9$, and 2.2. Find the shortest confidence interval for μ with each of the following three confidence coefficients: (a) 0.90 , (b) 0.95 , and (c) 0.99 .

7. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and unknown variance σ^2, and let the random variable L denote the length of the shortest confidence interval for μ that can be constructed from the observed values in the sample. Find the value of $E\left(L^2\right)$ for the following values of the sample size n and the confidence coefficient γ :
   a. $n=5, \gamma=0.95$
   d. $n=8, \gamma=0.90$
   b. $n=10, \gamma=0.95$
   e. $n=8, \gamma=0.95$
   c. $n=30, \gamma=0.95$
   f. $n=8, \gamma=0.99$

8. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and known variance σ^2. How large a random sample must be taken in order that there will be a confidence interval for μ with confidence coefficient 0.95 and length less than 0.01σ ?