Recitation 8

Instructions for Exercises 1 to 5 : In each of these exercises, assume that the random variables $X_1, \ldots, X_n$ form a random sample of size n from the distribution specified in that exercise, and show that the statistic T specified in the exercise is a sufficient statistic for the parameter.

1. The Bernoulli distribution with parameter p, which is unknown $(0 < p < 1)$; $T = \sum_{i=1}^{n} X_i$.

2. The geometric distribution with parameter p, which is unknown $(0 < p < 1)$; $T = \sum_{i=1}^{n} X_i$.

3. The normal distribution for which the mean μ is known and the variance $\sigma^2 > 0$ is unknown; $T = \sum_{i=1}^{n} (X_i - \mu)^2$.

4. The gamma distribution with parameters α and β, where the value of α is known and the value of β is unknown $(\beta > 0)$; $T = \bar{X}_n$.

5. The gamma distribution with parameters α and β, where the value of β is known and the value of α is unknown $(\alpha > 0)$; $T = \prod_{i=1}^{n} X_i$.

6. Consider a distribution for which the p.d.f. or the p.f. is $f(x \mid \theta)$, where the parameter θ is a k-dimensional vector belonging to some parameter space Ω. It is said that the family of distributions indexed by the values of θ in Ω is a k-parameter exponential family, or a k-parameter Koopman-Darmois family, if $f(x \mid \theta)$ can be written as follows for $\theta \in \Omega$ and all values of x :

$$f(x \mid \theta) = a(\theta) b(x) \exp\left[\sum_{i=1}^{k} c_i(\theta) d_i(x)\right].$$

   Here, a and $c_1, \ldots, c_k$ are arbitrary functions of θ, and b and $d_1, \ldots, d_k$ are arbitrary functions of x. Suppose now that $X_1, \ldots, X_n$ form a random sample from a distribution which belongs to a k-parameter exponential family of this type, and define the k statistics $T_1, \ldots, T_k$ as follows:

$$T_i = \sum_{j=1}^{n} d_i(X_j) \quad \text{for } i = 1, \ldots, k.$$

   Show that the statistics $T_1, \ldots, T_k$ are jointly sufficient statistics for θ.

7. Show that each of the following families of distributions is a two-parameter exponential family as defined in Exercise 6: a. The family of all normal distributions for which both the mean and the variance are unknown b. The family of all gamma distributions for which both α and β are unknown c. The family of all beta distributions for which both α and β are unknown