Parametric Statistics

Sampling Distributions of Estimators

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Lecture Summary

8.2 The χ^2 -distributions.

8.3 Joint Distribution of the Sample Mean and Variance.

MLE for μ, σ^2

Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Find the MLE for

- μ when σ^2 is known.
- σ^2 when μ is known.
- μ, σ^2 when they are both unknown.

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- μ when σ^2 is known. $\hat{\mu} = \overline{X}_n$
- σ^2 when μ is known. $\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$
- μ, σ^2 when they are both unknown. $\hat{\mu} = \overline{X}_n$, $\hat{\sigma_0^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$

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Bias?

The χ^2 distributions

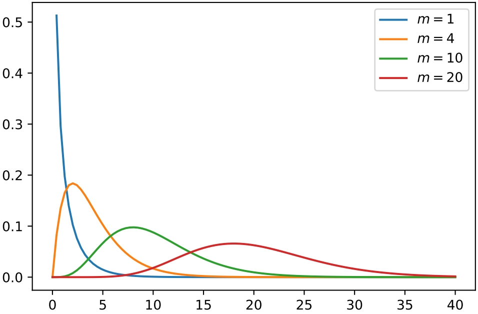
Definition

The χ^2 distribution with m degrees of freedom is the $Gamma(a = m/2, \beta = 1/2)$. The pdf is

$$f(x|m) = \frac{1}{2^{m/2}\Gamma(m/2)} x^{m/2-1} e^{-x/2}$$

If $X \sim \chi^2_m$ then

- $E(X) = m$
- $Var(X) = 2m$
- No closed form CDF, tabulated at the end of statistics books



Properties of the χ^2 distributions

Theorem (8.2.1)

Let $X_1, \ldots, X_n$ be independent random variables and $X_i \sim \chi^2_{m_i}$. Then

$$X_1 + \cdots + X_n \sim \chi^2_m$$

where $m = m_1 + \cdots + m_n$.

- follows from Theorem 5.7.7. and the MGF:

$$\psi(t) = \left(\frac{1}{1-2t}\right)^{m/2}, \quad t < \frac{1}{2}$$

Theorem (8.2.2)

If $X \sim \mathcal{N}(0,1)$, then $X^2 \sim \chi^2_1$

Properties of the χ^2 distributions

Corollary (8.2.1)

If the random variables $X_1, \ldots, X_n$ *i.i.d.,* $X_i \sim \mathcal{N}(0,1)$ *then*

$$X_1^2 + \cdots + X_n^2 \sim \chi_n^2$$

The χ_m^2 distribution is a sampling distribution related to the sample variance of a normal distribution:

- If $X_1, \ldots, X_n$ are i.i.d, $X_i \sim \mathcal{N}(\mu, \sigma^2)$, where μ is known and the MLE of σ^2 is

$$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

- Then

$$\frac{n\hat{\sigma}^2_{MLE}}{\sigma^2} \sim \chi_n^2$$

Sampling Distributions of the normal sample mean and variance

- Let $X_1, \ldots, X_n$ be a random sample from a $\mathcal{N}(\mu, \sigma^2)$ with unknown μ, σ^2.
- The sample mean and the sample variance are defined as

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

- They are the MLEs for μ and σ^2 in this setting.

Theorem (8.3.1)

Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{N}(\mu, \sigma^2)$. Then $\overline{X}_n$ and S_n are independent random variables and $\bar{X}_n \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$, $\frac{nS_n}{\sigma^2} \sim \chi^2_{n-1}$.

Sampling Distribution of the Normal sample mean and variance

Theorem (8.3.1)

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Then $\overline{X}_n$ and S_n are independent random variables and

$$\overline{X}_n \sim \mathcal{N}(\mu, \sigma^2/n),$$

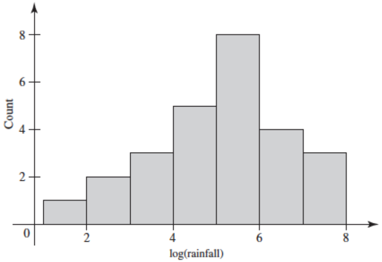
$$\frac{n}{\sigma^2} S_n = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \sim \chi^2_{n-1}$$

- Replacing μ by $\overline{X}_n$ results in one less degree of freedom.
- $\overline{X}_n$ and S_n are functions of the same random variables, but they are independent (this only happens when the random sample is drawn from a normal distribution).

Example

Rain from Seeded Clouds

- Simpson, Olsen, and Eden (1975).
- 26 clouds were seeded with silver nitrate to see if they produced more rain than unseeded clouds.
- Unseeded clouds produce mean rainfall of 4 (log scale).
- We are interested in how far the average log-rainfall of the seeded clouds $\hat{\mu}$ is from 4.



Example

How probable is it that we have overestimated the variance by more than 25%?

$$P(\hat{\sigma}^2 \leq 0.75\sigma^2) = P(\frac{26\hat{\sigma}^2}{\sigma^2} \leq 0.75 * 26) = 0.227$$

What is the smallest number of samples such that

$$\mathrm{P}\left(|\hat{\mu} - \mu| \leq \frac{1}{5}\sigma \quad , \quad |\hat{\sigma} - \sigma| \leq \frac{1}{5}\sigma\right) \geq \frac{1}{2}$$

Recap

- Today we discussed samples of normal distributions:
- MLE estimators of μ and σ^2 in different settings.
- Sampling distributions of these estimators.
- Sample mean and sample variance are independent!