Parametric Statistics-Recitation 7 (Solutions)

Exercise 1.

The data

$$\mathbf{x} = \{1, 16, 13, 9, 30, 6, 2, 21, 1\}$$

are an i.i.d. sample from a distribution with p.m.f:

$$\mathrm{P}_\theta(X = x) = (1-\theta)\theta^x, x \in= \{0, 1, 2, \ldots\}$$

for some $\theta \in (0, 1)$ [1]

(a) Find the likelihood function and MLE for θ.

(b) Find the method of moments estimator for θ.

(c) Using a $Beta(\alpha, \beta)$ prior for θ, find the posterior distribution $f(\theta|\mathbf{x})$.

Beta distribution: $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}$ where $\mathrm{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and Γ is the Gamma function.
Geometric distribution: $f(x) = (1-p)^x p$, $E[x] = \frac{1-p}{p}$

Solution.

(a) The likelihood function is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^{9} f(x_i|\theta) = (1-\theta)^9 \theta^{\sum_{i=1}^{9} x_i} = (1-\theta)^9\theta^{99}$$

In order to find the M.L.E we compute the log likelihood function $l(\theta) = 9log(1-\theta) + 99log\theta$ and solve $l'(\theta) = 0$ which gives us $\theta_{MLE} = \frac{11}{12}$.

(b) For the method of moments estimator for θ, i have only 1 parameter so the estimator is given by the equation

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ or } \frac{\theta}{1-\theta} = \frac{1}{9}99$$

and so $\theta_{MOM} = \frac{11}{12}$.

(c) If $\theta \sim Beta(a, b)$ then $\pi(\theta) = \frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1}$. So for the posterior $\pi(\theta|\mathbf{x})$ i know

$$\pi(\theta|\mathbf{x}) \propto \pi(\theta)f(\mathbf{x}|\theta) \propto \theta^{a+99-1}(a-\theta)^{b+9-1}$$

So $\theta|\mathbf{x} \sim Beta(a + 99, b + 9)$.

Exercise 2.

Suppose that a regular light bulb, a long-life light bulb, and an extra-long-life light bulb are being tested. The lifetime X_1 of the regular bulb has the exponential distribution with mean θ, the lifetime X_2 of the long-life bulb has the exponential distribution with mean 2θ, and the lifetime X_3 of the extra-long-life bulb has the exponential distribution with mean 3θ. a. Determine the M.L.E. of θ based on the observations X_1, X_2, and X_3. b. Let $\psi = 1/\theta$, and suppose that the prior distribution of ψ is the gamma distribution with parameters α and β. Determine the posterior distribution of ψ given X_1, X_2, and X_3.

[1] this is the Geometric distribution but parameterized in terms of the failure probability instead of the more usual success probability.

Solution.

The joint p.d.f of X_1, X_2, X_3 is

$$f(\mathbf{x}|\theta) = \frac{1}{\theta}e^{-\frac{x_1}{\theta}}\frac{1}{2\theta}e^{\frac{-x_2}{2\theta}}\frac{1}{3\theta}e^{-\frac{x_3}{3\theta}} = \frac{1}{6\theta^3}exp\left[-\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3}\right)\frac{1}{\theta}\right]$$

(a) By solving the equation $l'(\theta) = 0$ where $l(\theta) = logf(\mathbf{x}|\theta)$, we find that

$$\theta_{MLE} = \frac{x_1}{3} + \frac{x_2}{6} + \frac{x_3}{9}$$

(b) In terms of ψ, the joint p.d.f of X_1, X_2, X_3 is

$$f(\mathbf{x}|\psi) = \frac{\psi^3}{6}exp\left[-\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3}\right)\psi\right]$$

Since the prior p.d.f of ψ is $\pi(\psi) \propto \psi^{a-1}e^{-b\psi}$ it follows that the posterior p.d.f is

$$\pi(\psi|\mathbf{x}) \propto \pi(\psi)f(\mathbf{x}|\psi) \propto \psi^{a+2}exp\left[-\left(b + x_1 + \frac{x_2}{2} + \frac{x_3}{3}\right)\psi\right]$$

Hence $\psi|\mathbf{x} \sim Gamma(a+3, b + x_1 + \frac{x_2}{2} + \frac{x_3}{3})$.

Exercise 3.

Suppose that X_1 and X_2 are independent random variables, and that X_i has the normal distribution with mean $b_i\mu$ and variance σ_i^2 for $i = 1, 2$. Suppose also that b_1, b_2, σ_1^2, and σ_2^2 are known positive constants, and that μ is an unknown parameter. Determine the M.L.E. of μ based on X_1 and X_2.

Solution.

The joint p.d.f of X_1 and X_2 is

$$f(\mathbf{x}|\mu) = \frac{1}{\sqrt{2\pi\sigma_1\sigma_2}}exp\left[-\frac{(x_1 - b_1\mu)^2}{2\sigma_1^2} - \frac{(x_2 - b_2\mu)^2}{2\sigma_2^2}\right]$$

If we let $l(\mu) = logf(\mathbf{x}|\mu)$ and solve the equation $l'(\mu) = 0$ we get

$$\mu_{MLE} = \frac{\sigma_2^2 b_1 x_1 + \sigma_1^2 b_2 x_2}{\sigma_2^2 b_1^2 + \sigma_1^2 b_2^2}$$

Exercise 4.

Suppose that $X_1, \ldots, X_n$ form a random sample from a gamma distribution for which both parameters α and β are unknown. Find the M.L.E. of α/β.

Solution.

If we let $y = \sum_{i=1}^n x_i$, then the likelihood function is

$$f(\mathbf{x}|a, b) = \frac{b^{na}}{[\Gamma(a)]^n}\left[\prod_{i=1}^n x_i\right]^{a-1} e^{-by}$$

Now if we let $l(a, b) = logf(\mathbf{x}|a, b)$, then

$$l(a, b) = nalogb - nlog\Gamma(a) + (a-1)log\prod_{i=1}^n x_i - by$$

Hence $\frac{\delta l(a,b)}{db} = \frac{na}{b} - y$ and a, b must satisfy the equation $\frac{\delta l(a,b)}{db} = 0$ (as well the equation $\frac{\delta l(a,b)}{da} = 0$), it follows that $\frac{a}{b} = \frac{y}{n} = \bar{X}_n$.

Exercise 5.

Suppose that $X_1, \ldots, X_n$ follow a Normal distribution for which both parameters μ and σ^2 are unknown. Find the M.L.E. of μ and σ.

Solution.

Let $\theta = \sigma^2$. The joint p.d.f of $X_1, ..., X_n$ is

$$f(\mathbf{x}|\mu, \theta) = (2\pi\theta)^{-n/2} exp\left[-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\theta}\right]$$

Now let $l(\mu, \theta) = log f(\mathbf{x}|\mu, \theta)$. The M.L.E of μ and θ are given by the solution of the system

$$\frac{\delta l(\mu,\theta)}{d\mu} = 0 \text{ and } \frac{\delta l(\mu,\theta)}{d\theta} = 0$$

$$\sum_{i=1}^{n}\frac{x_i-\mu}{\theta} = 0 \text{ and } -\frac{n}{2}\frac{1}{2\pi\theta}2\pi + \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\theta^2} = 0$$

$$\sum_{i=1}^{n}(x_i-\mu) = 0 \text{ and } -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\sum_{i=1}^{n}x_i - n\mu = 0 \text{ and } -n + \frac{1}{\theta}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\mu_{MLE} = \bar{X_n} \text{ and } \theta_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{X_n})^2$$

So $\mu_{MLE} = \bar{X_n}$ and $\sigma_{MLE} = \sqrt{\theta_{MLE}}$.