Recitation 7

1. The data

$$\mathbf{x} = \{1, 16, 13, 9, 30, 6, 2, 21, 1\}$$

are an i.i.d. sample from a distribution with p.m.f:

$$\mathrm{P}_\theta(X = x) = (1 - \theta)\theta^x, x \in= \{0, 1, 2, \dots\}$$

for some $\theta \in (0, 1)$ [1]

   (a) Find the likelihood function and MLE for θ.

   (b) Find the method of moments estimator for θ.

   (c) Using a $Beta(\alpha, \beta)$ prior for θ, find the posterior distribution $f(\theta|\mathbf{x})$.

   Beta distribution: $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}$ where $\mathrm{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and Γ is the Gamma function.
   Geometric distribution: $f(x) = (1 - p)^x p$, $E[x] = \frac{1-p}{p}$

2. Suppose that a regular light bulb, a long-life light bulb, and an extra-long-life light bulb are being tested. The lifetime X_1 of the regular bulb has the exponential distribution with mean θ, the lifetime X_2 of the long-life bulb has the exponential distribution with mean 2θ, and the lifetime X_3 of the extra-long-life bulb has the exponential distribution with mean 3θ. a. Determine the M.L.E. of θ based on the observations X_1, X_2, and X_3. b. Let $\psi = 1/\theta$, and suppose that the prior distribution of ψ is the gamma distribution with parameters α and β. Determine the posterior distribution of ψ given X_1, X_2, and X_3.

3. Suppose that X_1 and X_2 are independent random variables, and that X_i has the normal distribution with mean $b_i\mu$ and variance σ_i^2 for $i = 1, 2$. Suppose also that b_1, b_2, σ_1^2, and σ_2^2 are known positive constants, and that μ is an unknown parameter. Determine the M.L.E. of μ based on X_1 and X_2.

4. Suppose that $X_1, \dots, X_n$ form a random sample from a gamma distribution for which both parameters α and β are unknown. Find the M.L.E. of α/β.

5. Suppose that $X_1, \dots, X_n$ follow a Normal distribution for which both parameters μ and σ^2 are unknown. Find the M.L.E. of μ and σ.

[1] this is the Geometric distribution but parameterized in terms of the failure probability instead of the more usual success probability.