Recitation 5

1. Suppose that X is a random variable for which

$$\Pr(X \geq 0) = 1 \text{ and } \Pr(X \geq 10) = 1/5.$$

   Prove that $E(X) \geq 2$.

2. Suppose that $X_1, \ldots, X_n$ form a random sample of size n from a distribution for which the mean is 6.5 and the variance is 4 . Determine how large the value of n must be in order for the following relation to be satisfied:

$$\Pr\left(6 \leq \bar{X}_n \leq 7\right) \geq 0.8.$$

3. Suppose that 30 percent of the items in a large manufactured lot are of poor quality. Suppose also that a random sample of n items is to be taken from the lot, and let Q_n denote the proportion of the items in the sample that are of poor quality. Find a value of n such that $\Pr\left(0.2 \leq Q_n \leq 0.4\right) \geq 0.75$ by using (a) the Chebyshev inequality and (b) the tables of the binomial distribution at the end of this book.

4. It is said that a sequence of random variables Z_1, $Z_2, \ldots$ converges to a constant b in quadratic mean if

$$\lim_{n \to \infty} E\left[\left(Z_n - b\right)^2\right] = 0.$$

   Show that Eq. (6.2.17) is satisfied if and only if

$$\lim_{n \to \infty} E\left(Z_n\right) = b \text{ and } \lim_{n \to \infty} \operatorname{Var}\left(Z_n\right) = 0.$$

   Hint: Use Exercise 5 of Sec. 4.3 in the DeGroot and Schervish book.

5. Suppose that the distribution of the number of defects on any given bolt of cloth is the Poisson distribution with mean 5 , and the number of defects on each bolt is counted for a random sample of 125 bolts. Determine the probability that the average number of defects per bolt in the sample will be less than 5.5 .

6. A random sample of n items is to be taken from a distribution with mean μ and standard deviation σ.
   (a) Use the Chebyshev inequality to determine the smallest number of items n that must be taken in order to satisfy the following relation:

$$\Pr\left(\left|\bar{X}_n - \mu\right| \leq \frac{\sigma}{4}\right) \geq 0.99.$$

   (b) Use the central limit theorem to determine the smallest number of items n that must be taken in order to satisfy the relation in part (a) approximately.

7. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with unknown mean θ and variance σ^2. Assuming that $\theta \neq 0$, determine the asymptotic distribution of $\bar{X}_n^3$.