Parametric Statistics

Special Distributions

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Lecture Summary

5.2 The Bernoulli and Binomial Distributions

5.4 The Poisson Distributions

5.5 The Negative Binomial Distributions

5.6 The Normal Distributions

5.7 The Gamma Distributions (only the Exponential Distribution)

Bernoulli distributions

Definition

A random variable X has the Bernoulli distribution with parameter $p (0 \le p \le 1)$ if X can take only the values 0 and 1 and the probabilities are

$$\Pr(X = 1) = p \text{ and } \Pr(X = 0) = 1 - p.$$

The p.f. of X can be written as follows:

$$f(x \mid p) = \begin{cases} p^x (1-p)^{1-x} & \text{for } x = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Parameter space: $p \in [0, 1]$.
- $E(X) = p, Var(X) = p(1 - p)$.
- MGF: $\psi(t) = E(e^{tX}) = pe^t + (1 - p)$.
- CDF:

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - p, & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Bernoulli trials

Bernoulli Trials/Process

If the random variables in a finite or infinite sequence $X_1, X_2, \ldots$ are i.i.d, and if each random variable X_i has the Bernoulli distribution with parameter p, then it is said that $X_1, X_2, \ldots$ are Bernoulli trials with parameter p.

An infinite sequence of Bernoulli trials is also called a Bernoulli process.

Binomial distributions

Definition

A random variable X has the binomial distribution with parameters n and p if X has a discrete distribution for which the p.f. is as follows:

$$f(x \mid n, p) = \begin{cases} \left(\begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

In this distribution, n must be a positive integer, and p must lie in the interval $0 \leq p \leq 1$.

- Parameter space: n positive integer, $p \in [0, 1]$.
- $E(X) = np, Var(X) = np(1-p)$.
- CDF:$F(x) = (n-k) \left(\begin{array}{c} n \\ k \end{array} \right) \int_0^{1-p} t^{n-k-1}(1-t)^k dt.$

Poisson Distributions

The Poisson distribution is useful for modeling uncertainty in events in a fixed time period.

Examples:

- How many calls in a call center in one hour?
- How many busses pass while you wait at the bus stop for 10 min?
- How many customers will enter a store in 15 minutes?

Poisson Distributions

Definition

Let $\lambda > 0$. A random variable X has the Poisson distribution with mean λ if the p.f. of X is as follows:

$$f(x \mid \lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & \text{for } x = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

- Parameter space: $\lambda \in [0, \infty)$.
- $E(X) = \lambda, Var(X) = \lambda$
- MGF: $\psi(t) = e^{\lambda\left(e^t - 1\right)}$
- CDF: $e^{-\lambda} \sum_{j=0}^{\lfloor k \rfloor} \frac{\lambda^j}{j!}$

Properties of the Poisson

Theorem (Sum of Poissons is a Poisson.)

Let $X_1, \dots X_k$ *are independent and if* X_i *has the Poisson distribution with mean* $\lambda_i (i = 1, \dots, k)$*, then the sum* $X_1 + \cdots + X_k$ *has the Poisson distribution with mean* $\lambda_1 + \cdots + \lambda_k$*.*

Theorem (Approximation to the Binomial)

For each integer n *and each* $0 < p < 1$*, let* $f(x|n, p)$ *denote the pf of the Binomial distribution with parameters* n *and* p*, and let* $f(x|\lambda)$ *denote the pf of the Poisson distribution with mean* λ*. Let* $\{p_n\}_1^\infty$ *be a sequence of numbers between* 0 *and* 1 *such that* $lim_{n \to \infty} = \lambda$*. Then*

$$lim_{n \to \infty} f_{X_n}(x|n, p_n) = f(x|\lambda)$$

When the value of n is large, and the value of p is very small, the Poisson with mean np is a good approximation for the Binomial with parameters n and p.

Example

The number of emails that I get in a weekday can be modeled by a Poisson distribution with an average of 0.2 emails per minute.

- What is the probability that I get no emails in an interval of length 5 minutes?
- What is the probability that I get more than 3 emails in an interval of length 10 minutes?

Solution

1. Let X be the number of emails that I get in the 5 -minute interval. Then, by the assumption X is a Poisson random variable with parameter $\lambda = 5(0.2) = 1$,

$$P(X=0) = P_X(0) = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-1}\cdot 1}{1} = \frac{1}{e} \approx 0.3679$$

2. Let Y be the number of emails that I get in the 10 -minute interval. Then by the assumptior Y is a Poisson random variable with parameter $\lambda = 10(0.2) = 2$,

$$\begin{aligned}
P(Y>3) &= 1 - P(Y \le 3) \\
&= 1 - (P_Y(0) + P_Y(1) + P_Y(2) + P_Y(3)) \\
&= 1 - e^{-\lambda} - \frac{e^{-\lambda}\lambda}{1!} - \frac{e^{-\lambda}\lambda^2}{2!} - \frac{e^{-\lambda}\lambda^3}{3!} \\
&= 1 - e^{-2} - \frac{2e^{-2}}{1} - \frac{4e^{-2}}{2} - \frac{8e^{-2}}{6} \\
&= 1 - e^{-2}\left(1 + 2 + 2 + \frac{8}{6}\right) \\
&= 1 - \frac{19}{3e^2} \approx 0.1429
\end{aligned}$$

Negative Binomial distributions

Definition

A random variable X has the negative binomial distribution with parameters r and $p(r = 1, 2, \dots$ and $0 < p < 1)$ if X has a discrete distribution for which the p.f. $f(x \mid r, p)$ is:

$$f(x \mid r, p) = \begin{cases} \begin{pmatrix} r + x - 1 \\ x \end{pmatrix} p^r (1-p)^x & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that an infinite sequence of Bernoulli trials with probability of success p are available. The number X of failures that occur before the $r=$ th success follows a negative binomial with parameters r, p.

- $E(X) = \frac{r(1-p)}{p}$
- $Var(X) = \frac{r(1-p)}{p^2}$
- MGF: $\psi(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$

Geometric distributions

Definition

A random variable X has the geometric distribution with parameter $p (0 < p < 1)$ if X has a discrete distribution for which the p.f. $f(x \mid 1, p)$ is as follows:

$$f(x \mid 1, p) = \begin{cases} p(1-p)^x & \text{for } x = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

- X = number of failures before the first success.
- Parameter space $p \in (0, 1)$.
- $E(X) = \frac{1-p}{p}$
- $Var(X) = \frac{1-p}{p^2}$
- MGF: $\psi(t) = \frac{p}{1-(1-p)e^t}$

Properties of Geometric distributions

Sum of Geometric is Negative Binomial

If $X_1, ..., X_r$ are i.i.d. and each $X_i \sim Geometric(p)$ then $X = X_1 + \cdots + X_r \sim NegBinomial(r, p)$.

Geometric distributions are memoryless:

Let X have the geometric distribution with parameter p, and let $k \geq 0$. Then for every integer $t \geq 0$,

$$P(X = k + t | X \geq k) = P(X = t).$$

The Exponential Distributions

Definition

Let $\beta > 0$. A random variable X has the exponential distribution with parameter β if X has a continuous distribution with the p.d.f.

$$f(x \mid \beta) = \begin{cases} \beta e^{-\beta x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

A comparison of the p.d.f.'s for gamma and exponential distributions makes the following result obvious.

- $E(X) = \frac{1}{\beta}$
- $Var(X) = \frac{1}{\beta^2}$
- MGF: $\psi(t) = \frac{\beta}{\beta - t}$ for $t < \beta$

Properties of the Exponential Distributions

Exponential distributions are memoryless

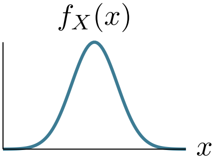
Let X have the exponential distribution with parameter β, and let $t > 0$. Then for every number $h > 0$,

$$P(X \geq t + h | X \geq t) = P(X \geq h)$$

Minimum of exponentials is exponential

Let $X_1, X_2, \ldots, X_n$ each follow an exponential distribution with parameter β. Then the distribution of $Y = min\{X_1, \ldots, X_n\}$ will be the exponential distribution with parameter $n\beta$.

The Normal Distribution



Definition

A random variable X has the normal distribution with mean μ and variance $\sigma^2 (-\infty < \mu < \infty$ and $\sigma > 0)$ if X has a continuous distribution with the following p.d.f.:

$$f\left(x \mid \mu, \sigma^2\right) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \text{ for } -\infty < x < \infty$$

- MGF: $\psi(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$ for $-\infty < t < \infty$

Properties of normal distributions

Linear Transformations of normals are normal

If X has the normal distribution with mean μ and variance σ^2 and if $Y = aX + b$, where a and b are given constants and $a \neq 0$, then Y has the normal distribution with mean $a\mu+b$ and variance $a^2\sigma^2$.

Linear Combinations of Independent normals are normal

If the random variables $X_1, \dots, X_k$ are independent and if X_i has the normal distribution with mean μ_i and variance $\sigma_i^2 (i = 1, \dots, k)$, then the sum $X_1 + \cdots + X_k$ has the normal distribution with mean $\mu_1 + \cdots + \mu_k$ and variance $\sigma_1^2 + \cdots + \sigma_k^2$.

- Assume $X_1, \dots X_n$ are a random sample from $N(\mu, \sigma^2)$.
- What is the distribution of the sample mean, $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$?

The Standard Normal

Definition

The normal distribution with mean 0 and variance 1 is called the standard normal distribution. The p.d.f. of the standard normal distribution is usually denoted by the symbol ϕ, and the c.d.f. is denoted by the symbol Φ. Thus,

$$\phi(x) = f(x \mid 0, 1) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2}x^2\right) \text{ for } -\infty < x < \infty$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(u) du \text{ for } -\infty < x < \infty,$$

where the symbol u is used as a dummy variable of integration.

Computing probabilities of the normal distribution

Consequences of Symmetry.

For all x and all $0 < p < 1$,

$$\Phi(-x) = 1 - \Phi(x) \text{ and } \Phi^{-1}(p) = -\Phi^{-1}(1-p).$$

Converting Normal Distributions to Standard Normal.

Let X have the normal distribution with mean μ and variance σ^2. Let F be the c.d.f. of X. Then $Z = (X - \mu)/\sigma$ has the standard normal distribution, and, for all x and all $0 < p < 1$,

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$F^{-1}(p) = \mu + \sigma\Phi^{-1}(p).$$

Computing probabilities of the normal distribution

Suppose that X has the normal distribution with mean 5 and standard deviation 2 . What is $\Pr(1 < X < 8)$?

Computing probabilities of the normal distribution

Suppose that X has the normal distribution with mean 5 and standard deviation 2 . What is $\Pr(1 < X < 8)$?
If we let $Z = (X - 5)/2$, then Z will have the standard normal distribution and

$$\Pr(1 < X < 8) = \Pr\left(\frac{1-5}{2} < \frac{X-5}{2} < \frac{8-5}{2}\right) = \Pr(-2 < Z < 1.5).$$

Furthermore,

$$\begin{aligned}\Pr(-2 < Z < 1.5) &= \Pr(Z < 1.5) - \Pr(Z \leq -2) \\ &= \Phi(1.5) - \Phi(-2) \\ &= \Phi(1.5) - [1 - \Phi(2)].\end{aligned}$$

From the table at the end of this book, it is found that $\Phi(1.5) = 0.9332$ and $\Phi(2) = 0.9773$. Therefore,

$$\Pr(1 < X < 8) = 0.9105.$$