Parametric Statistics-Recitation 2 (Solutions)

Exercise 1.

Suppose that a random variable X has a discrete distribution with the following p.f.:

$$f(x) = \begin{cases} \frac{c}{2^x} & x = 0, 1, 2, ... \\ 0 & otherwise \end{cases}$$

Find the value of the constant c.

Solution

It must be $\sum_{x=0}^{\infty} f(x) = 1$. So $\sum_{x=0}^{\infty} \frac{c}{2^x} = 1$. That means

$$c = \frac{1}{\sum_{x=0}^{\infty} 2^{-x}}$$

But

$$\sum_{x=0}^{\infty} 2^{-x} = \frac{1}{1 - \frac{1}{2}} = 2$$

And so $c = \frac{1}{2}$.

Exercise 2.

A civil engineer is studying a left-turn lane that is long enough to hold seven cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x + 1)(8 - x)$ for $x = 0, ..., 7$ (the possible values of X).
(a) Find the p.f. of X.
(b) Find the probability that X will be at least 5.

Solution

(a) The p.f. of X is $f(x) = c(x + 1)(8 - x)$ for $x = 0, ..., 7$. It must be $\sum_{x=0}^{7} f(x) = 1$. Also

$$\sum_{x=0}^{7} (x + 1)(8 - x) = 120$$

That means $c = \frac{1}{120}$ and $f(x) = \frac{(x+1)(8-x)}{120}$.

(b) $P(X \geq 5) = \sum_{x=5}^{7} \frac{(x+1)(8-x)}{120} = \frac{1}{3}$

Exercise 3.

Suppose that the p.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} cx^2 & 1 \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

(a) Find the value of the constant c and sketch the p.d.f.
(b) Find the value of $Pr(X > \frac{3}{2})$.

Solution

(a) Since $f(x)$ is the p.d.f of X we know that $\int_{-\infty}^{\infty} f(x)dx = 1$. That means

$$\int_1^2 cx^2 dx = 1$$

and so $c = \frac{3}{7}$.

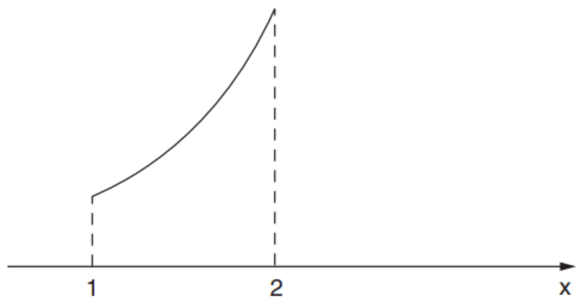


Figure 1: P.D.F of X

(b) $Pr(X > \frac{3}{2}) = \int_{\frac{3}{2}}^{2} \frac{3x^2}{7} dx = \frac{37}{56}$

Exercise 4.

Suppose that the p.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} ce^{-2x} & x > 0 \\ 0 & otherwise \end{cases}$$

(a) Find the value of the constant c and sketch the p.d.f.
(b) Find the value of $Pr(1 < X < 2)$.

Solution

(a) Since $f(x)$ is the p.d.f of X we know that $\int_{-\infty}^{\infty} f(x)dx = 1$. That means

$$\int_0^{\infty} ce^{-2x} dx = 1$$

and so $c = 2$.

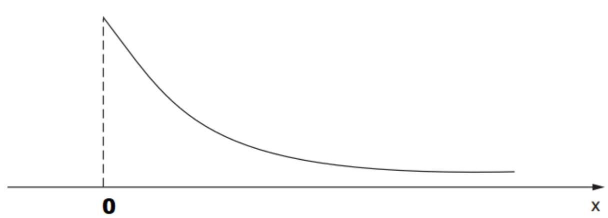
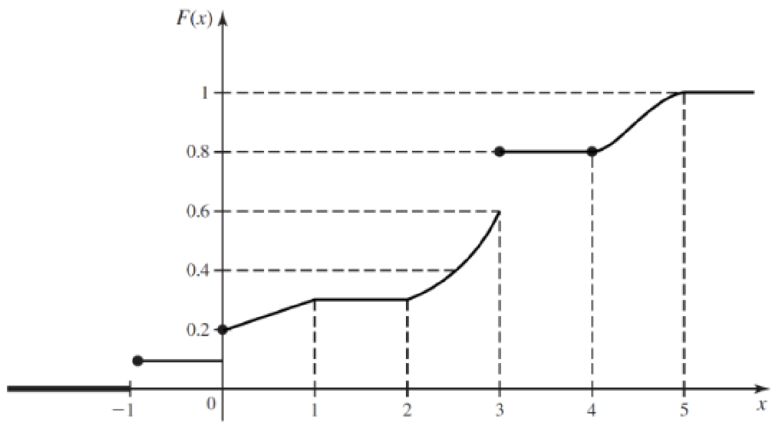


Figure 2: P.D.F of X

(b) $Pr(1 < X < 2) = \int_1^2 2e^{-2x} dx = e^{-2} - e^{-4}$.

Exercise 5.

Suppose that the c.d.f. F of a random variable X is as sketched below. Find each of the following probabilities:



a. $\Pr(X = -1)$
b. $\Pr(X < 0)$
c. $\Pr(X \leq 0)$
d. $\Pr(X = 1)$
e. $\Pr(0 < X \leq 3)$
f. $\Pr(0 < X < 3)$
g. $\Pr(0 \leq X \leq 3)$
h. $\Pr(1 < X \leq 2)$
i. $\Pr(1 \leq X \leq 2)$
j. $\Pr(X > 5)$
k. $\Pr(X \geq 5)$
I. $\Pr(3 \leq X \leq 4)$

Figure 3: C.D.F of X

Solution

(a) $F(-1) - F(-1^-) = 0.1$
(b) $F(0^-) = 0.1$
(c) $F(0) = 0.2$
(d) $F(1) - F(1^-) = 0$
(e) $F(3) - F(0) = 0.6$
(f) $F(3^-) - F(0) = 0.4$
(g) $F(3) - F(0^-) = 0.7$
(h) $F(2) - F(1) = 0$
(i) $F(2) - F(1^-) = 0$
(j) $1 - F(5) = 0$
(k) $1 - F(5^-) = 0$
(I) $F(4) - F(3^-) = 0.2$

Exercise 6.

Suppose that the c.d.f. of a random variable X is as follows:

$$F(x) = \begin{cases} e^{x-3} & x \leq 3 \\ 1 & x > 3 \end{cases}$$

Find and sketch the p.d.f. of X.

Solution

We know that

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} e^{x-3} & x < 3 \\ 0 & x > 3 \end{cases}$$

The value $x = 3$ is irrelevant.

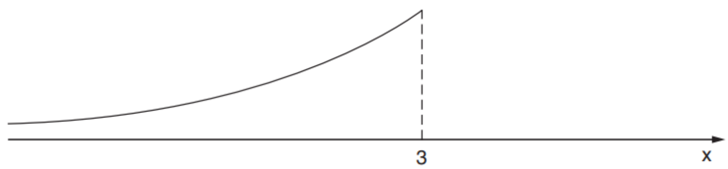


Figure 4: P.D.F of X

Exercise 7.

Suppose that X has the p.d.f.

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & otherwuise \end{cases}$$

Find and sketch the c.d.f. or X.

Solution

Since $f(x) = 0$ for $x \leq 0$ and for $x \geq 1$, the c.d.f $F(x)$ will take the value 0 for $x \leq 0$ and the value 1 for $x \geq 1$. Between 0 and 1, we compute $F(x)$ by integrating the p.d.f. So for $0 < x < 1$ we have that

$$F(x) = \int_0^x 2y dy = x^2$$

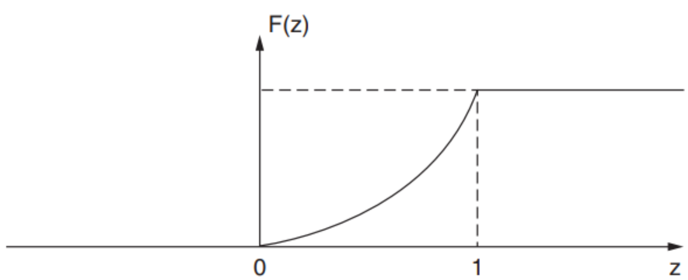


Figure 5: C.D.F of X