Parametric Statistics-Recitation 1 (Solutions)

Exercise 1.

If 50 percent of the families in a certain city subscribe to the morning newspaper, 65 percent of the families subscribe to the afternoon newspaper, and 85 percent of the families subscribe to at least one of the two newspapers, what percentage of the families subscribe to both newspapers?

Solution.

Let A be the event that a randomly selected subscribes to the morning paper and B the event that a randomly selected family subscribes to the afternoon paper. We know that $Pr(A) = 0.5, Pr(B) = 0.65, Pr(A \cup B) = 0.85$. So $Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B) = 0.3$

Exercise 2.

For two arbitrary events A and B, prove that

$$\operatorname{Pr}(A) = \operatorname{Pr}(A \cap B) + \operatorname{Pr}\left(A \cap B^c\right).$$

Solution.

We know by theorem that $A = (A \cap B) \cup (A \cap B^c)$. Clearly the events $(A \cap B)$ and $(A \cap B^c)$ are disjoint so

$$\operatorname{Pr}(A) = \operatorname{Pr}(A \cap B) + \operatorname{Pr}\left(A \cap B^c\right).$$

Exercise 3.

A point (x, y) is to be selected from the square S containing all points (x, y) such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Suppose that the probability that the selected point will belong to each specified subset of S is equal to the area of that subset. Find the probability of each of the following subsets: (a) the subset of points such that $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \geq \frac{1}{4}$; (b) the subset of points such that $\frac{1}{2} < x + y < \frac{3}{2}$; (c) the subset of points such that $y \leq 1 - x^2$; (d) the subset of points such that $x = y$.

Solution.

(a) The set of points for which $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 < \frac{1}{4}$ is the interior of a circle that is contained in the unit square (center $(\frac{1}{2}, \frac{1}{2})$ and radius $\frac{1}{2}$). The area of this circle is $\frac{\pi}{4}$ and so the desired area is $1 - \frac{\pi}{4}$.

(b) We need the area of region between the two lines $y = \frac{1}{2} - x$ and $y = \frac{3}{2} - x$. The remaining area is the union of two right triangles with height and base both equal to $\frac{1}{2}$. Each triangle has area $\frac{1}{8}$ so the region between the two lines has area $1 - \frac{2}{8}$.

(c) The desired area is $\int_0^1 (1 - x^2)dx = \frac{2}{3}$.

(d) The area of a line is 0 and so the desired probability.

Exercise 4.

Let the sample space S be the unit cube, i.e. $0 < x < 1, 0 < y < 1$ and $0 < z < 1$. For $A \subset S$ we define $P(A) = \text{Volume}(A)$.

(a) Show that P is a probability function.

(b) Find the probability of $A = \{(x, y, z) : 0 < x < y < 1 \text{ and } z \leq y \exp(-x)\}$. Hint: You find the volume of A by integrating $y \exp(-x)$ over the right values of x and y.

Solution.

(a)

- $P(A) = Volume(A) \geq 0$ for every $A \subset S$
- $P(S) = Volume(S) = 1$
- The volume of the union of every 2 disjoint subsets of S is the sum of their areas (and so we can prove it for 2 or more).

(b) $P(A) = Volume(A) = \int_0^1 \int_x^1 ye^{-x} dy dx \approxeq 0.23576$

Exercise 5.

Which of the following two numbers is larger: $\binom{93}{30}$ or $\binom{93}{31}$?

Solution.

The ratio of $\binom{93}{30}$ to $\binom{93}{31}$ is $\frac{31}{63} < 1$ so $\binom{93}{31}$ is bigger.

Exercise 6.

Which of the following two numbers is larger: $\binom{93}{30}$ or $\binom{93}{63}$?

Solution.

Since 93=63+30 , the two numbers are the same.

Exercise 7.

A box contains 24 light bulbs, of which four are defective. If a person selects four bulbs from the box at random, without replacement, what is the probability that all four bulbs will be defective?

Solution.

All the possible 4 selects out of 24 light bulbs are $\binom{24}{4}$. There is only 1 selection of 4 defective so the desired probability is 1 over $\binom{24}{4}$.

Exercise 8.

If six dice are rolled, what is the probability that each of the six different numbers will appear exactly once?

Solution.

With 6 rolls there are 6^6 possible outcomes. The outcomes with all different rolls are 6!. So the desired probability is $\frac{6!}{6^6}$.

Exercise 9.

Each time a shopper purchases a tube of toothpaste, he chooses either brand A or brand B. Suppose that for each purchase after the first, the probability is 1/3 that he will choose the same brand that he chose on his preceding purchase and the probability is 2/3 that he will switch brands. If he is equally likely to choose either brand A or brand B on his first purchase, what is the probability that both his first and second purchases will be brand A and both his third and fourth purchases will be brand B?

Solution.

Let A_i stand for the event that the shopper purchases brand A on his ith purchase. for $i = 1, 2, ...$ Similarly, let B_i be the event that he purchases brand B on the ith purchase. Then we have

$$Pr(A_1) = \frac{1}{2}$$
$$Pr(A_2|A_1) = \frac{1}{3}$$
$$Pr(B_3|A_2 \cap A_1) = \frac{2}{3}$$
$$Pr(B_4|B_3 \cap A_2 \cap A_1) = \frac{1}{3}$$

The desired probability is the product of these probabilities, which is $\frac{1}{27}$.

Exercise 10.

A machine produces defective parts with three different probabilities depending on its state of repair. If the machine is in good working order, it produces defective parts with probability 0.02. If it is wearing down, it produces defective parts with probability 0.1. If it needs maintenance, it produces defective parts with probability 0.3 . The probability that the machine is in good working order is 0.8, the probability that it is wearing down is 0.1, and the probability that it needs maintenance is 0.1. Compute the probability that a randomly selected part will be defective.

Solution.

We define 4 events as following:
A: A part is defective.
B_1: The machine is in good working order.
B_2: The machine is wearing down.
B_3: The machine needs maintance.
We know that

$$Pr(B_1) = 0.8, Pr(B_2) = 0.1, Pr(B_3) = 0.1, Pr(A|B_1) = 0.02, Pr(A|B_2) = 0.1, Pr(A|B_3) = 0.3$$

The desired probability is $Pr(A)$ which is computed by the law of total probability

$$Pr(A) = \sum_{i=1}^{3} Pr(B_i)Pr(A|B_i) = 0.8 \cdot 0.02 + 0.1 \cdot 0.1 + 0.1 \cdot 0.3 = 0.056$$

Exercise 11.

Suppose that a person rolls two balanced dice three times in succession. Determine the probability that on each of the three rolls, the sum of the two numbers that appear will be 7.

Solution.

Since the three rolls are independent we just have to find the probability that in the first roll of the 2 dice the sum of the two numbers will be 7. There are 6 ways out of 36 possible were the 2 numbers of the dice can have sum 7 : (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) and so the probability is $\frac{1}{6}$. Finally the desired probability is $(\frac{1}{6})^3$.

Exercise 12.

Suppose that A, B, and C are three independent events such that Pr(A) = 1/4, Pr(B) = 1/3 and Pr(C) = 1/2. (a) Determine the probability that none of these three events will occur. (b) Determine the probability that exactly one of these three events will occur.

Solution.

We will use the following 2 propositions:

- If A and B are independent then A^c and B^c are independent.
- If A and B are independent then A and B^c are independent.

(a) The desired probability is

$$Pr(A^c \cap B^c \cap C^c) = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4}$$

(b) The desired probability is

$$Pr(A \cap B^c \cap C^c) + Pr(A^c \cap B \cap C^c) + Pr(A^c \cap B^c \cap C) = ... = \frac{11}{24}$$

Exercise 13.

Suppose that the probability that any particle emitted by a radioactive material will penetrate a certain shield is 0.01. If 10 particles are emitted, what is the probability that exactly one of the particles will penetrate the shield?

Solution.

We want 1 particle to penetrate the shield and 9 not to. That can happen with $\binom{10}{1}$ ways, that is 10 ways. So the desired probability is $10 \cdot 0.01 \cdot 0.99^9$.

Exercise 14.

Consider again the conditions of the previous exercise. If 10 particles are emitted, what is the probability that at least one of the particles will penetrate the shield?

Solution.

The probability that none of the 10 particles will penetrate the shield is 0.99^{10}. Therefore, the probability that at least one particle will penetrate the shield is $1 - 0.99^{10}$.

Exercise 15.

In a certain city, 30 percent of the people are Conservatives, 50 percent are Liberals, and 20 percent are Independents. Records show that in a particular election, 65 percent of the Conservatives voted, 82 percent of the Liberals voted, and 50 percent of the Independents voted. If a person in the city is selected at random and it is learned that she did not vote in the last election, what is the probability that she is a Liberal?

Solution.

We define 4 events as following:
C: A person is Conservative.
L: A person is Liberal.
I: A person is Independent.
V: A person voted.

The desired probability is $Pr(L|V^c)$. By applying the Bayes formula we have that:

$$Pr(L|V^c) = \frac{Pr(V^c|L)Pr(L)}{Pr(V^c|C)Pr(C) + Pr(V^c|L)Pr(L) + Pr(V^c|I)Pr(I)} =$$

$$= \frac{0.18 \cdot 0.5}{0.35 \cdot 0.3 + 0.18 \cdot 0.5 + 0.5 \cdot 0.2} = \frac{18}{59}$$

Exercise 16.

Suppose that when a machine is adjusted properly, 50 percent of the items produced by it are of high quality and the other 50 percent are of medium quality. Suppose, however, that the machine is improperly adjusted during 10 percent of the time and that, under these conditions, 25 percent of the items produced by it are of high quality and 75 percent are of medium quality.

(a) Suppose that five items produced by the machine at a certain time are selected at random and inspected. If four of these items are of high quality and one item is of medium quality, what is the probability that the machine was adjusted properly at that time?
(b) Suppose that one additional item, which was produced by the machine at the same time as the other five items, is selected and found to be of medium quality. What is the new posterior probability that the machine was adjusted properly?

Solution.

We define the following events:
P: The machine is adjusted properly.
I: The machine is adjusted improperly.
A: 4 out of 5 inspected items are of high quality.

(a) The desired probability is $Pr(P|A)$.

$$Pr(P|A) = \frac{Pr(A|P)Pr(P)}{Pr(A|P)Pr(P) + Pr(A|I)Pr(I)} =$$

$$= \frac{0.9 \cdot \binom{5}{4} \cdot 0.5^5}{0.9 \cdot \binom{5}{4} \cdot 0.5^5 + 0.1 \cdot \binom{5}{4} \cdot 0.25^4 \cdot 0.75} = \frac{96}{97}$$

(b) Our new prior probabilities are calculated in part (a). So $Pr(P) = \frac{96}{97}$ and $Pr(I) = \frac{1}{97}$. Let B the event that the additional item is of medium quality. The desired probability is

$$Pr(P|B) = \frac{Pr(P)Pr(B|P)}{Pr(P)Pr(B|A) + Pr(I)Pr(B|I)} =$$

$$= \frac{\frac{96}{97} \cdot \frac{1}{2}}{\frac{96}{97} \cdot \frac{1}{2} + \frac{1}{97} \cdot \frac{3}{4}} = \frac{64}{65}$$