Parametric Statistics

Random Variables

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Lecture Summary

3.1 Discrete Random Variables

3.2 Continuous Random Variables

3.3 The Cumulative Distribution Function

3.4 Bivariate Distributions

3.5 Marginal Distributions

Some slides from MIT open courseware

Last time

- Two events are called independent when knowing the value of one doesn't influence the probability of the value of the other.
- The conditional probability of A given B denotes the probability of event A in a world where B has occurred.
- We can use the multiplication to compute conditional, marginal and joint probabilities.
- Bayes rule connects $P(A|B)$ and $P(B|A)$. These two are confused but they are not the same.

Why is Bayes Rule so important?

- V: vaccinated
- H: hospitalized.
- $P(H|V) = 0.01$
- $P(H|V^c) = 0.2$
- Three different possibilities: $P(V) = 0.8, 0.5, 0.95$

Let' s use Bayes rule to compute $P(V|H)$ for all three cases.

$P(V)$	$P(H|V)$	$P(H|V^c)$	$P(V|H)$
0.5	0.01	0.2	0.0476
0.8	0.01	0.2	0.16667
0.95	0.01	0.2	0.4872

Random Variables

Random Variable

A random variable is a mapping $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each outcome ω.

Example

Consider the experiment of flipping a coin 10 times, and let $X(\omega)$ denote the number of heads in the outcome ω. For example, if $\omega = HHHTHHTTHT$, $X(\omega) = 6$.

Random Variables

- Why do we need random variables? (easier to work with than original sample space).
- A random variable is NOT a variable (in the algebraic sense).
- A random variable takes a specific value AFTER the experiment is conducted.

Notation

- Letter near the end of the alphabet since it is a variable in the context of the experiment.
- Capital letter to distinguish from algebraic variable.
- Lower case denotes a specific value of the random variable.

Random Variable: Definition

- An assignment of a value (number) to every possible outcome.
- A real-valued function of the sample space:

$$f : \Omega \to \mathbb{R}$$

- Can take discrete or continuous values.
- Discrete RVs have probability mass functions.
- Continuous RVs have probability density functions.

Discrete Random Variables

Discrete Distribution/Random Variable

We say that a random variable X has a discrete distribution or that X is a discrete random variable if X can take only a finite number k of different values $x_1, \ldots, x_k$ or, at most, an infinite sequence of different values $x_1, x_2, \ldots$.

Probability Function/p.f./Support.

If a random variable X has a discrete distribution, the probability function (abbreviated p.f.) of X is defined as the function f such that for every real number x,

$$f(x) = \Pr(X = x).$$

The closure of the set $\{x : f(x) > 0\}$ is called the support of (the distribution of) X.

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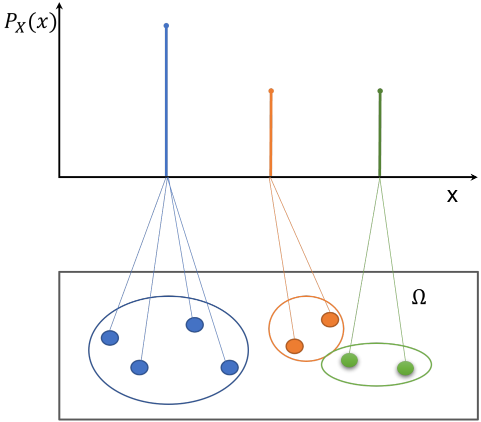
- Also known as the probability mass function.

Probability (mass) function



If $C \subset \mathbb{R}$:

$$\Pr(X \in C) = \sum_{x_i \in C} f(x_i),$$

If $x_1, x_2, \ldots$ includes all the possible values of X, then

$$\sum_{i=1}^{\infty} f(x_i) = 1$$

Bernoulli Distribution

Example: Coin Toss

- You toss a biased coin.
- $P(Heads) = p$
- $\Omega = \{Heads, Tails\}$
- Define $X(Heads) = 1$, $X(Tails) = 0$
- $f(x) =$

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Distribution

$$f(x) = \begin{cases} p^x(1-p)^{1-x} & \text{for } x = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Discrete Uniform Distribution

Example: Lottery

- Select a three digit number (leading 0 s allowed).
- Each digit is chosen at random from a separate urn with 10 balls, each with a different digit.
- $\Omega = (i_1, i_2, i_3)$ where $i_j \in \{0, \dots, 9\}$ for $j = 1, 2, 3$
- If $\omega = (i_1, i_2, i_3)$, define $X(\omega) = 100i_1 + 10i_2 + i_3$. For example, $X(0, 1, 5) = 15$
- $f(x) =$

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Distribution

$$f(x) = \begin{cases} \frac{1}{b-a+1} & \text{for } x = a, \ldots, b, \\ 0 & \text{otherwise.} \end{cases}$$

Continuous Random Variables

Continuous Distribution/Random Variable

We say that a random variable X has a continuous distribution or that X is a continuous random variable if there exists a non-negative function f, defined on the real line, such that for every interval of real numbers (bounded or unbounded), the probability that X takes a value in the interval is the integral of f over the interval.

Probability Function/p.f./Support.

If X has a continuous distribution, the function f is called the probability density function (abbreviated p.d.f.) of X. The closure of the set $\{x : f(x) > 0\}$ is called the support of (the distribution of) X.

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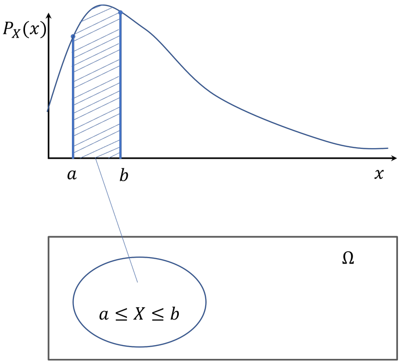

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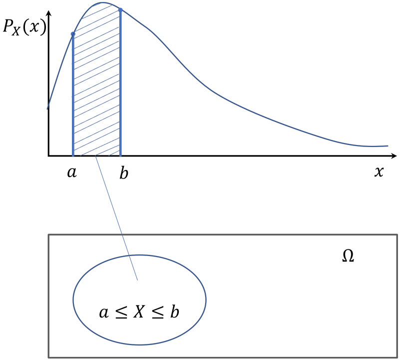

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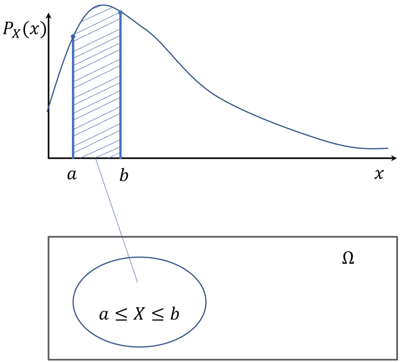

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- $P(X = a) = \int_a^a f_X(x)dx = 0$

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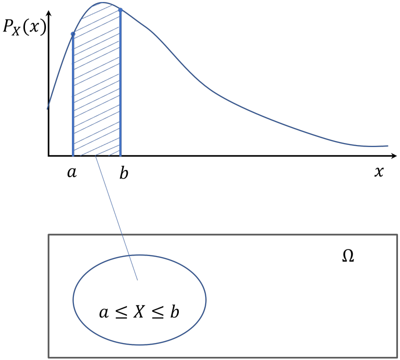

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- $\int_{-\infty}^{\infty} f_X(x) = 1$

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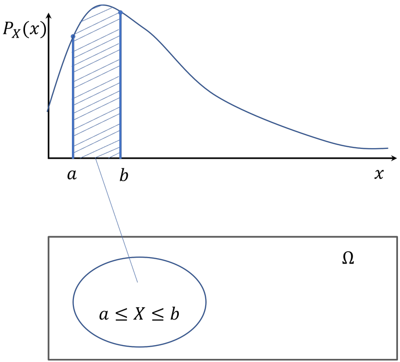

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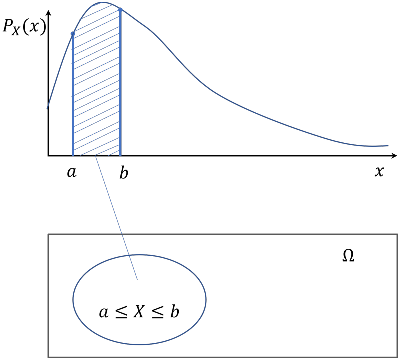
Density is not probability!

Continuous Random Variables



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Density is not probability!

$$P(a \leq X \leq a + \epsilon = \int_a^{a+\epsilon} f_X(x)dx = f(a)\epsilon$$

$$P(b \leq X \leq b + \epsilon = \int_b^{b+\epsilon} f_X(x)dx = f(b)\epsilon$$

Continuous Uniform Distribution

Example: Weather

- Temperature Forecasts announce high and low temperature forecasts as integer numbers of degrees.
- In reality, the degrees are rounded up to the closest integer.
- Suppose that the forecaster announces a high temperature of y.
- X: the actual high temperature.
- X was equally likely to be any number in the interval from $y - 1/2$ to $y + 1/2$.

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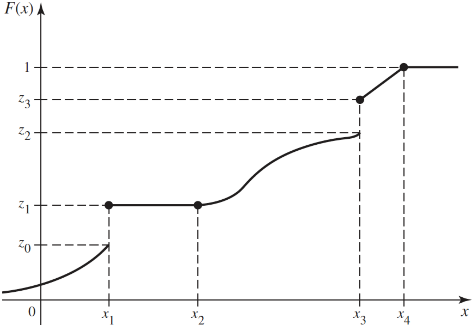
Distribution on the interval $[a, b]$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{otherwise} \end{cases}$$

Cumulative Distribution Function (CDF)

$$F_X(x) = P(X \leq x) = \begin{cases} \int_{-\infty}^{x} f_X(t)dt, & \text{if } X \text{ is continuous} \\ \sum_{k \leq x} p_X(x) & \text{if } X \text{ is discrete} \end{cases}$$



Properties of the CDF

- Nondecreasing. The function $F(x)$ is nondecreasing as x increases; that is, if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.

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- Limits at $\pm\infty$. $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

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- Limits at $\pm\infty$. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.
- Continuity from the Right. A c.d.f. is always continuous from the right; that is, $F(x) = F(x^+)$ at every point x.

Properties of the CDF

- For every value x,

$$\Pr(X > x) = 1 - F(x).$$

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- For all values x_1 and x_2 such that $x_1 < x_2$,

$$\Pr\left(x_1 < X \leq x_2\right) = F\left(x_2\right) - F\left(x_1\right).$$

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- **Proof.** Let $A = \left\{x_1 < X \leq x_2\right\}, B = \{X \leq x_1\}$, and $C = \{X \leq x_2\}$. A and B are disjoint, and their union is C, so

$$\Pr\left(x_1 < X \leq x_2\right) + \Pr\left(X \leq x_1\right) = \Pr\left(X \leq x_2\right).$$

Properties of the CDF

Let X have a continuous distribution, and let $f(x)$ and $F(x)$ denote its p.d.f. and the c.d.f., respectively. Then F is continuous at every x,

$$F(x) = \int_{-\infty}^{x} f(t)dt,$$

and

$$\frac{dF(x)}{dx} = f(x),$$

at all x such that f is continuous.

Quantiles/Percentiles

Fair Bets

Suppose that X is the number of points of the winning team on NBA basketball game, and X has c.d.f. F. Suppose that we want to place an even-money bet on X as follows: If $X \leq x_0$, we win one dollar and if $X > x_0$ we lose one dollar.
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How can you make this bet fair? $\Pr(X \leq x_0) = \Pr(X > x_0) = 1/2$.

Quantiles/Percentiles.

Let X be a random variable with c.d.f. F. For each p strictly between 0 and 1 , define $F^{-1}(p)$ to be the smallest value x such that $F(x) \geq p$. Then $F^{-1}(p)$ is called the p quantile of X or the $100p$ percentile of X. The function F^{-1} defined here on the open interval $(0, 1)$ is called the quantile function of X.

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- Example: Compute the quantile function for the uniform distribution.

Joint Probability Mass Function

y \ x	0	1	2	3
3	$1/20$	$2/20$	$2/20$	$2/20$
2	$2/20$	$3/20$	$1/20$	$1/20$
1		$2/20$	$3/20$	
0	$1/20$			

$$f(x, y) = P(X = x, Y = y)$$

$$\sum_{all(x,y)} f(x, y) = 1 \text{ (still a probability mass function)}$$

$$\text{Marginal Probability:} f(x) = \sum_{y} f(x, y) \text{ (sum over all possible y)}$$

Joint Probability Mass Function

y \ x	0	1	2	3
3	1/20	2/20	2/20	2/20
2	2/20	3/20	1/20	1/20
1		2/20	3/20	
0	1/20			

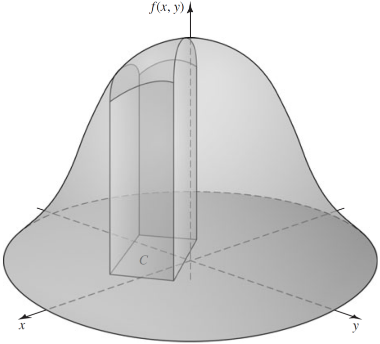
$$f(x, y) = P(X = x, Y = y)$$

$$\sum_{all(x,y)} f(x, y) = 1 \text{ (still a probability mass function)}$$

Marginal Probability: $f(x) = \sum_{y} f(x, y)$ (sum over all possible y)

What is $P(X \geq 2, Y \geq 2)$?

Joint Probability Density Function



Joint p.d.f.:

$$f(x, y) \geq 0, \text{everywhere}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$\Pr[(X, Y) \in C] = \int_C \int f(x, y) dx dy$$

Mixed Distributions

Joint p.f./p.d.f

Let X and Y be random variables such that X is discrete and Y is continuous. Suppose that there is a function $f(x, y)$ defined on the xy-plane such that, for every pair A and B of subsets of the real numbers,

$$\Pr(X \in A \text{ and } Y \in B) = \int_B \sum_{x \in A} f(x, y) dy,$$

if the integral exists. Then the function f is called the joint p.f./p.d.f. of X and Y.

Joint (Cumulative) Distribution Function/c.d.f.

The joint distribution function or joint cumulative distribution function (joint c.d.f.) of two random variables X and Y is defined as the function F such that for all values of x and $y(-\infty < x < \infty$ and $-\infty < y < \infty)$

$$F(x, y) = \Pr(X \leq x \text{ and } Y \leq y).$$

Recap

- Random variables are functions from the sample space to the real line.
- Random variables can be discrete or continuous.
- Discrete distributions have probability mass functions.
- Continuous distributions have probability density functions.
- Both distributions can be described with the cumulative distribution function.
- The quantile function is the inverse of the CDF, for continuous distributions.
- Pairs of random variables have joint distributions.