Parametric Statistics

Conditional Probability

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Last Time

- Probability spaces formalize uncertainty.
- Sample spaces describe possible outcomes of a random experiment.
- Events are subsets of the sample space.
- Probability measure defines how probable events are.
- We need to define them on sets of events that are *measurable*, i.e., σ-algebras.
- Probability measures need to satisfy the *axioms of probability.*
- In discrete uniform sample spaces, we can compute probabilities by counting (a) the number of outcomes in our event (b) the number of outcomes in the sample space.

Today

2.1 Conditional Probability

2.3 Independent Events

2.4 Bayes Rule

- SKIP: 2.4 The Gambler's Ruin Problem

Reminder: Probability Measure

We want to assign a real number $\mathbb{P}(A)$ to every element A of a $\sigma-$algebra $\mathcal{A}$ to $[0, 1]$ which represents how likely event A is to occur. This is called the probability of A.

- Axiom 1: $\mathbb{P}(A) \geq 0$ for every A
- Axiom 2: $\mathbb{P}(\Omega) = 1$
- Axiom 3: If $A_1, A_2, A_3, \ldots$ are events and $A_i \cap A_j = \emptyset$ for all $i \neq j$ then
$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

The triplet $(\Omega, \mathcal{A}, \mathbb{P})$ is called a **probability space**.

Counting in a uniform (simple) probability space

In discrete sample spaces, if all outcomes of an experiments are considered equally likely, then for each event A,

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

Conditional Probability

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- You see that the first number is 15, but then broadcast stops. What are your chances of winning now?
- We just described the **conditional probability** $P(A|B)$

Conditional Probability

Definition (Conditional Probability of A given B)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The probability of event A in the universe (sample space) where event B has already happened.

Conditional Probabilities are probabilities!

- $P(A) + P(A^c) = 1$
- $P(A \cup C) = P(A) + P(C) - P(A \cap C)$
- $P(\Omega) = 1$

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- $P(\Omega|B) = P(B|B) = 1$

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$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

$$P(\bigcap_{i=1}^{n} A_i) = \prod_{k=1}^{n} P(A_i | \bigcap_{j=1}^{k-1} A_j)$$

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- If I tell you that the first toss is "heads" it doesn't change your belief about what will happen in the second toss.

Definition

Two events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

We denote this as

$$A \perp\!\!\!\perp B$$

A set of events $\{A_i : i \in I\}$ are independent if

$$\mathbb{P}(\cap_{i \in J} A_i) = \prod \mathbb{P}(A_i)$$

for every finite subset J of I.

Independence as conditional probability

- P(Second toss is heads|first toss is heads) = 1/2 = P(Second toss is heads)
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$
- If I tell you that B has happened, this does not change your belief about how likely A is.

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Are disjoint events independent?

Conditional Independence

Just Independence in the new universe:

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Notation Change!

Instead of $A \cap B$ use A, B

$$P(A, B|C) = P(A|C)P(B|C)$$

- If A, B are conditionally independent given C, then

$$P(A|B, C) =$$

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$$P(A|B, C) = P(A|C)$$

Conditional Independence $\neq$ Independence

You roll a die twice.

- A: first die is 1.
- B: second die is 1.
- C: Sum of the two dice is 5.

Are A and B independent? Are they independent given C?

A box has a regular and a fake coin with two heads. You choose one at random and toss it twice.

- A: first toss is Heads.
- B: second toss is Heads.
- C: The regular coin is selected.

Are A and B independent? Are they independent given C?

Multiplication Rule Example #2

Example

- You go past a clinic that gives tests for a rare (1 in 1000 people) disease. The clinic tells you that the tests are 95% accurate, so
  - $P(T^{+}|D+) = 0.95.$
  - $P(T^{-}|D^{-}) = 0.95$
- What is the probability that you test positive and you don't have the disease?
- Your test comes out positive. What is the probability that you have the disease?

Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Let's prove it:

Bayes Rule

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Let's prove it:

$$P(A, B) = P(A)P(B \mid A)$$

On the other hand, the probability of A and B is also equal to the probability of B times the probability of A given B.

$$P(A, B) = P(B)P(A \mid B)$$

Equating the two yields:

$$P(B)P(A, B) = P(A)P(B \mid A)$$

and thus

$$P(A, B) = P(A)\frac{P(B \mid A)}{P(B)}$$

Why is Bayes Rule so important?

Very often, people confuse $P(A|B)$ and $P(B|A)$. These can be VERY different.

Think about it:

You read in the paper:

"Half of the people hospitalized with covid-19 are fully vaccinated".

Do you think that getting the vaccine lowers your chances of getting hospitalized?

Why is Bayes Rule so important?

- V: vaccinated
- H: hospitalized.
- $P(H|V) = 0.01$
- $P(H|V^c) = 0.2$
- Three different possibilities: $P(V) = 0.8, 0.5, 0.99$

Let' s use Bayes rule to compute $P(V|H)$ for all three cases.

Review

- Two events are called independent when knowing the value of one doesn't influence the probability of the value of the other.
- The conditional probability of A given B denotes the probability of event A in a world where B has occurred.
- We can use the multiplication to compute conditional, marginal and joint probabilities.
- Bayes rule connects $P(A|B)$ and $P(B|A)$. These two are confused but they are not the same.

Recitation Exercises

Section	Exercises
2.1	4, 14
2.2	2, 12
2.3	5, 6