Parametric Statistics

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Lecture Summary

9.5 The t-test

9.6 Comparing the means of two normal distributions

9.7 The F distributions

Uniformly Most Powerful Tests

$$H_0 : \theta \in \Omega_0 \quad \text{vS} \quad H_1 : \theta \in \Omega_1$$

- A test δ^* is a uniformly most powerful test at level α_0 if for any other level α_0 test δ

$$\pi(\theta \mid \delta) \leq \pi\left(\theta \mid \delta^*\right) \quad \text{for all } \theta \in \Omega_1$$

I.o.w: It has the lowest probability of type II error of any test, uniformly for all $\theta \in \Omega_1$.

- We control the probability of type I error by setting the level (size) of the test low. We then want to control the probability of type II error.
- If $\pi\left(\theta \mid \delta^*\right)$ is high for all $\theta \in \Omega_1$, the test is often called "powerful"
- In a large class of problems (the distribution has a "monotone likelihood ratio") we can find a uniformly most powerful test for one-sided hypotheses (Ch. 9.3).

The t-Test

- The t-Test is a test for hypotheses concerning the mean parameter in the normal distribution when the variance is also unknown.
- The test is based on the t distribution

The setup for the next few slides:

- Let $X_1, \ldots, X_n$ be i.i.d. $N\left(\mu, \sigma^2\right)$ and consider the hypotheses

$$H_0 : \mu \leq \mu_0 \quad \text{vs.} \quad H_1 : \mu > \mu_0$$

The parameter space here is $-\infty < \mu < \infty$ and $\sigma^2 > 0$, i.e.

$$\Omega = (-\infty, \infty) \times (0, \infty)$$

And

$$\Omega_0 = (-\infty, \mu_0] \times (0, \infty) \text{ and } \Omega_1 = (\mu_0, \infty) \times (0, \infty)$$

The one-sided t-Test

- Let

$$U = \frac{\sqrt{n}\left(\bar{X}_n - \mu_0\right)}{\sigma'} \quad \text{where } \sigma' = \left(\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\right)^{1/2}$$

- If $\mu = \mu_0$ then U has the t distribution
- Tests based on U are called t tests

The one-sided t-Test

- Let T_n^{-1} be the quantile function of the t_n distribution

Theorem (Theorem 9.5.1)

The test δ that rejects H_0 in (1) if $U \geq T_{n-1}^{-1}(1-\alpha_0)$ has size α_0 and a power function with the following properties
(i) $\pi\left(\mu_0, \sigma^2 \mid \delta\right) = \alpha_0$
(ii) $\pi\left(\mu, \sigma^2 \mid \delta\right) < \alpha_0$ *for* $\mu < \mu_0$
(iii) $\pi\left(\mu, \sigma^2 \mid \delta\right) > \alpha_0$ *for* $\mu > \mu_0$
(iv) $\pi\left(\mu, \sigma^2 \mid \delta\right) \to 0$ *as* $\mu \to -\infty$
(v) $\pi\left(\mu, \sigma^2 \mid \delta\right) \to 1$ *as* $\mu \to \infty$

The complete power function

To calculate the power function $\pi\left(\mu, \sigma^2 \mid \delta\right)$ exactly we need the non-central t_m distributions:

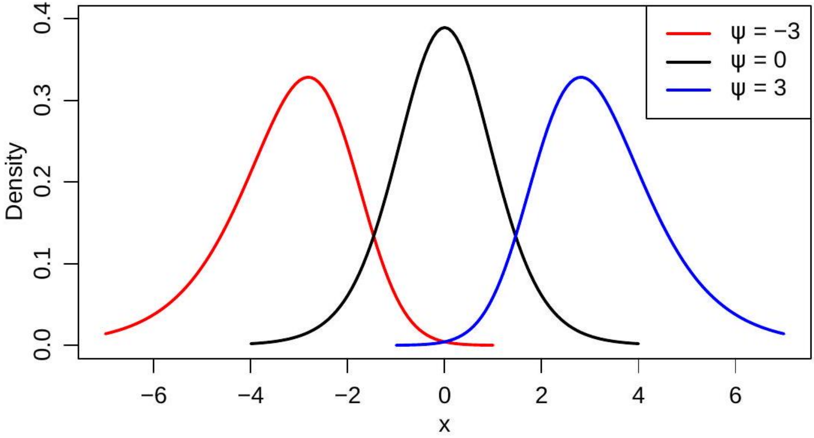
Definition

Let $W \sim N(\psi, 1)$ and $Y \sim \chi^2_m$ be independent. The distribution of

$$X = \frac{W}{(Y/m)^{1/2}}$$

is called the non-central t distribution with m degrees of freedom and non-centrality parameter ψ

Non-central t_m distribution



The complete power function

Theorem (Theorem 9.5.3)

U has the non-central t_{n-1} distribution with non-centrality parameter $\psi = \sqrt{n}\left(\mu - \mu_0\right)/\sigma$

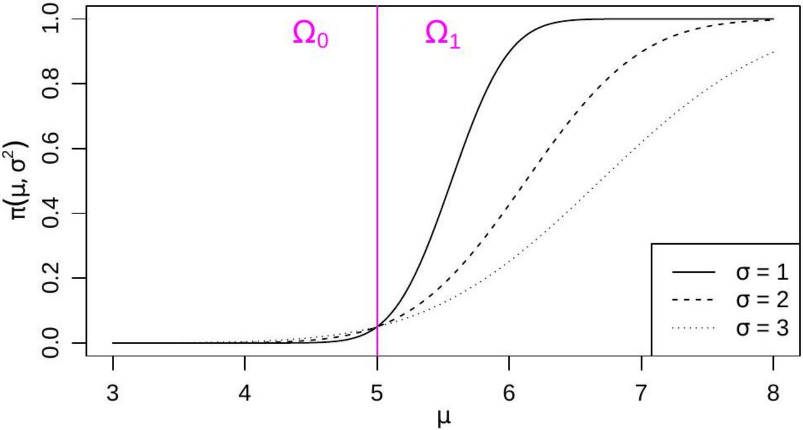
The power function of the t-test that rejects H_0 in (1) if $U \geq T_{n-1}^{-1}\left(1 - \alpha_0\right) = c_1$ is

$$\pi\left(\mu, \sigma^2 \mid \delta\right) = 1 - T_{n-1}\left(c_1 \mid \psi\right)$$

- Can use the R function $1 - \mathrm{pt}(\mathrm{q} = \mathrm{c1}, \mathrm{df} = \mathrm{n} - 1, \mathrm{ncp} = \mathrm{psi})$

Power function for the one-sided t-test

Example: $n = 10, \mu_0 = 5, \alpha_0 = 0.05$
Power function for the size $0.05\mathrm{t}-\mathrm{test}$, with $\mathrm{n}=10$



Note that the power function is a function of both σ^2 and μ

p-value for the one-sided t-Test

Theorem (9.5.2:p-values for t tests)

Let $H_0 : \mu \leq \mu_0$ vs. $H_1 : \mu > \mu_0$. Let δ be the one sided t-test for H_0, H_1. Let u be the observed value of U. The p-value is $1 - T_{n-1}(u)$.

Example: Acid Concentration in Cheese (Example 8.5.4)

- Have a random sample of $n = 10$ lactic acid measurements from cheese, assumed to be from a normal distribution with unknown mean and variance.
- Observed: $\bar{x}_n = 1.379$ and $\sigma' = 0.3277$
- Perform the level $\alpha_0 = 0.05$ t-test of the hypotheses

$$H_0 : \mu \leq 1.2 \quad \text{vs} \quad H_1 : \mu > 1.2$$

- Compute the p-value

The other one-sided t-Test

- Now consider the hypotheses

$$H_0 : \mu \geq \mu_0 \quad \text{vS.} \quad H_1 : \mu < \mu_0$$

Corollary (9.5.1)

The test δ that rejects H_0 if $U \leq T_{n-1}^{-1}(\alpha_0)$ has size α_0 and a power function with the following properties
(i) $\pi\left(\mu_0, \sigma^2 \mid \delta\right) = \alpha_0$
(ii) $\pi\left(\mu, \sigma^2 \mid \delta\right) > \alpha_0$ *for* $\mu < \mu_0$
(iii) $\pi\left(\mu, \sigma^2 \mid \delta\right) < \alpha_0$ *for* $\mu > \mu_0$
(iv) $\pi\left(\mu, \sigma^2 \mid \delta\right) \to 1$ *as* $\mu \to -\infty$
(v) $\pi\left(\mu, \sigma^2 \mid \delta\right) \to 0$ *as* $\mu \to \infty$

Power function and p-value for the other one-sided t-Test

Theorem (9.5.2: p-values for t Tests)

Let u be the observed value of U. The p-value is $T_{n-1}(u)$.

Theorem (Theorem 9.5.3)

U has the non-central t_{n-1} distribution with non-centrality parameter $\psi = \sqrt{n}\,(\mu - \mu_0)\,/\sigma$.

The power function of the t-test that rejects H_0 in (2) if $U \leq T_{n-1}^{-1}(\alpha_0) = c_2$ is

$$\pi\left(\mu, \sigma^2 \mid \delta\right) = T_{n-1}\left(c_2 \mid \psi\right)$$

Two-sided t-test

Consider now the test with a two-sided alternative hypothesis:

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0$$

- Let δ be the test that rejects H_0 iff $|U| \geq T_{n-1}^{-1}(1 - \alpha_0/2) = c$
- Then δ is a size α_0 test
- The power function is

$$\pi\left(\mu, \sigma^2 \mid \delta\right) = T_{n-1}(-c \mid \psi) + 1 - T_{n-1}(c \mid \psi)$$

- If u is the observed value of U then the p-value is $2\left(1 - T_{n-1}(|u|)\right)$

The t test is a likelihood ratio test (see p. 583 - 585 in the book)

The paired t-test

Sometimes we are measuring the same variable under two different conditions

- National Transportation Safety Board crash test dummy experiment:
- For each car, place
  - One dummy in the driver's seat.
  - One dummy in the passenger's seat.
  - Measure the head injuries for each dummy
- You want to compare: On average, which seat suffers the most injuries.
- Take $X_1, \ldots, X_n$ to be the difference for each car: Head injury in the driver's side - head injury in the passenger's side.
- Test $H_0 : \mu \geq 0$ vs $H_1 : \mu < 0$.

The two-sample t-test

Comparing the means of two populations

- $X_1, \ldots, X_m$ i.i.d. $N\left(\mu_1, \sigma^2\right)$ and
- $Y_1, \ldots, Y_n$ i.i.d. $N\left(\mu_2, \sigma^2\right)$
- The variance is the same for both samples, but unknown

We are interested in testing one of these hypotheses:
a) $H_0 : \mu_1 \leq \mu_2$ vs. $H_1 : \mu_1 > \mu_2$
b) $H_0 : \mu_1 \geq \mu_2$ vS. $H_1 : \mu_1 < \mu_2$
c) $H_0 : \mu_1 = \mu_2$ vS. $H_1 : \mu_1 \neq \mu_2$
Power function is now a function of 3 parameters: $\pi\left(\mu_1, \mu_2, \sigma^2 \mid \delta\right)$

Two-sample t statistic

Let $\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} x_i$ and $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$

$$S_X^2 = \sum_{i=1}^{m} \left(X_i - \bar{X}_m\right)^2 \quad \text{and} \quad S_Y^2 = \sum_{i=1}^{n} \left(Y_i - \bar{Y}_n\right)^2$$

$$U = \frac{\sqrt{m+n-2}\left(\bar{X}_m - \bar{Y}_n\right)}{\left(\frac{1}{m}+\frac{1}{n}\right)^{1/2}\left(S_X^2 + S_Y^2\right)^{1/2}}$$

- Theorem 9.6.1: If $\mu_1 = \mu_2$ then $U \sim t_{m+n-2}$
- Theorem 9.6.4: For any μ_1 and μ_2, U has the non-central t_{m+n-2} distribution with non-centrality parameter

$$\psi = \frac{\mu_1 - \mu_2}{\sigma(1/m + 1/n)^{1/2}}$$

Two-sample t test - summary

a) $H_0 : \mu_1 \leq \mu_2$ vs. $H_1 : \mu_1 > \mu_2$

- Level α_0 test: Reject H_0 iff $U \geq T^{-1}_{m+n-2}(1-\alpha_0)$
- p-value: $1 - T_{m+n-2}(u)$

b) $H_0 : \mu_1 \geq \mu_2$ vs. $H_1 : \mu_1 < \mu_2$

- Level α_0 test: Reject H_0 iff $U \leq T^{-1}_{m+n-2}(\alpha_0)$
- p-value: $T_{m+n-2}(u)$

c) $H_0 : \mu_1 = \mu_2$ vS. $H_1 : \mu_1 \neq \mu_2$

- Level α_0 test: Reject H_0 iff $|U| \geq T^{-1}_{m+n-2}(1-\alpha_0/2)$
- p-value: $2\left(1 - T_{m+n-2}(|u|)\right)$

The two-sample t-test is a likelihood ratio test (see p. 592)

Two-sample t test - unequal variances

- We can extend the two sample t-test to a problem where the variances of the X_i 's and Y_j 's are not equal but the ratio of them is known, i.e. $\sigma_1^2 = k\sigma_2^2 -$ Not very practical

In general, the problem where the variances are not equal is very hard.

- Proposed test-statistics do not have known distribution, but approximations have been obtained
- Example: The Welch statistic

$$V = \frac{\bar{X}_m - \bar{Y}_n}{\left(\frac{S_X^2}{m(m-1)} + \frac{S_Y^2}{n(n-1)}\right)^{1/2}}$$

can be approximated by a t distribution

- Example: The distribution of the likelihood ratio statistic can be approximated by the χ_1^2 distribution if the sample size is large enough

F-distributions

- In light of the previous slide, it would be nice to have a test of whether the variances in the two normal populations are equal $\rightarrow$ need the $F_{m,n}$ distributions

Definition

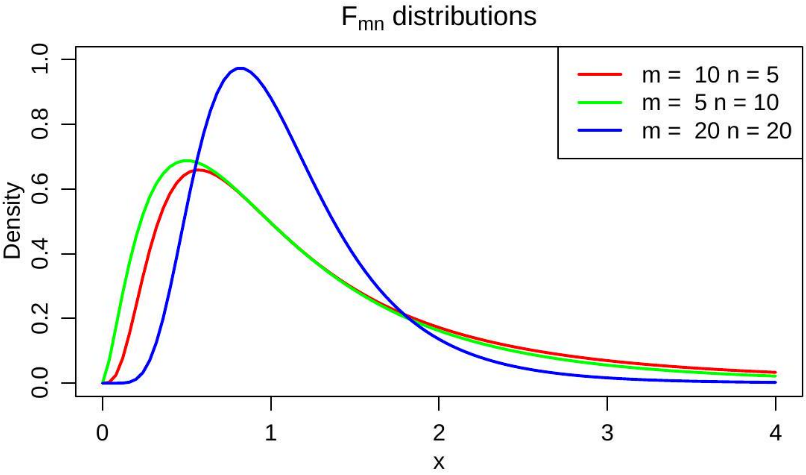
$F_{m,n}$-distributions Let $Y \sim \chi^2_m$ and $W \sim \chi^2_n$ be independent. The distribution of

$$X = \frac{Y/m}{W/n} = \frac{nY}{mW}$$

is called the F distribution with m and n degrees of freedom
The pdf of the $F_{m,n}$ distribution is

$$f(x) = \frac{\Gamma((m+n)/2)m^{m/2}n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \frac{x^{m/2-1}}{(mx+n)^{(m+n)/2}} \quad x > 0$$

F-distributions



- The 0.95 and 0.975 quantiles of the $F_{m,n}$ distribution is tabulated in the back of the book for a few combinations of m and n

Properties of F distributions

Theorem 9.7.2: Two properties

(i) If $X \sim F_{m,n}$ then $1/X \sim F_{n,m}$

(ii) If $Y \sim t_n$ then $Y^2 \sim F_{1,n}$

Comparing the variances of two populations

- $X_1, \ldots, X_m$ i.i.d. $N\left(\mu_1, \sigma_1^2\right)$ and
- $Y_1, \ldots, Y_n$ i.i.d. $N\left(\mu_2, \sigma_2^2\right)$
- All four parameters unknown

Consider the hypotheses:

$$H_0 : \sigma_1^2 \le \sigma_2^2 \quad \text{vs} \quad H_1 : \sigma_1^2 > \sigma_2^2$$

and the test that rejects H_0 if $V \ge c$, where

$$V = \frac{S_X^2/(m-1)}{S_Y^2/(n-1)}$$

This test is called an F-test

- $\frac{\sigma_2^2}{\sigma_1^2} V \sim F_{m-1,n-1}$
- If $\sigma_1^2 = \sigma_2^2$ then $V \sim F_{m-1,n-1}$

The F test

Let c be the $1-\alpha_0$ quantile of the F distribution with $m-1$ and $n-1$ degrees of freedom, and let $G_{m-1,n-1}$ be the c.d.f. of that F distribution. Let δ be test that rejects H_0 in when $V \geq c$. The power function $\pi\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \mid \delta\right)$ satisfies the following properties:

- $\pi\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \mid \delta\right) = 1 - G_{m-1,n-1}\left(\frac{\sigma_2^2}{\sigma_1^2} c\right)$,
- $\pi\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \mid \delta\right) = \alpha_0$ when $\sigma_1^2 = \sigma_2^2$
- $\pi\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \mid \delta\right) < \alpha_0$ when $\sigma_1^2 < \sigma_2^2$,
- $\pi\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \mid \delta\right) > \alpha_0$ when $\sigma_1^2 > \sigma_2^2$,
- $\pi\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \mid \delta\right) \to 0$ as $\sigma_1^2/\sigma_2^2 \to 0$,
- $\pi\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \mid \delta\right) \to 1$ as $\sigma_1^2/\sigma_2^2 \to \infty$.

The test δ has level α_0. The p-value when $V = v$ is observed equals $1 - G_{m-1,n-1}(v)$

Comparing the variances of two populations

- $X_1, \ldots, X_m$ i.i.d. $N\left(\mu_1, \sigma_1^2\right)$ and
- $Y_1, \ldots, Y_n$ i.i.d. $N\left(\mu_2, \sigma_2^2\right)$
- All four parameters unknown

Consider the hypotheses:

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \text{vs} \quad H_1 : \sigma_1^2 \neq \sigma_2^2$$

and the test that rejects H_0 if $V \geq c$, where

$$V = \frac{S_X^2/(m-1)}{S_Y^2/(n-1)}$$

This test is called an F-test

- $\frac{\sigma_2^2}{\sigma_1^2} V \sim F_{m-1,n-1}$
- If $\sigma_1^2 = \sigma_2^2$ then $V \sim F_{m-1,n-1}$

The two-sided F test

If we want to compare:

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \text{vs.} \quad H_1 : \sigma_1^2 \neq \sigma_2^2$$

- We will reject H_0 if $V \leq c_1$ or $V \geq c_2$.
- Typically we choose c_1, c_2 such that $P(V \leq c_1) = P(V \geq c_2) = \alpha_0/2$.
- Then, p-value $2 * min\{1 - G_{m-1,n-1}(v), G_{m-1,n-1}(v)\}$, where v is the observed value of V.