Lecture Summary

6.2 The central limit theorem

Convergence of Random Variables

Let $X_1, X_2, \ldots$ be a sequence of random variables, let X be a random variable.

Convergence in Probability

$X_1, X_2, \ldots$ converges in probability to X if

$$\forall \epsilon > 0, lim_{n \to \infty} P(|X_n - X| \geq \epsilon) = 0$$

Convergence in Distribution

$X_1, X_2, \ldots$ converges in distribution to X if

$$\forall \epsilon > 0, lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

for all x where F_X is continuous.

Convergence of Random Variables

Let $X_1, X_2, \ldots$ be a sequence of random variables, let X be a random variable.

Almost Sure Convergence

$X_1, X_2, \ldots$ converges almost surely to X if

$$\forall \epsilon > 0, P(lim_{n \to \infty} |X_n - X| \geq \epsilon) = 0$$

- Almost sure convergence implies convergence in probability.
- Convergence in probability implies convergence in distribution.
- Convergence in distribution does not imply similarity of the random variables, just their distributions.

Central Limit Theorem

Theorem (Central Limit Theorem)

If the random variables $X_1, \ldots, X_n$ form a random sample of size n from a given distribution with mean μ and variance σ^2 ($0 < \sigma^2 < \infty$), then for each fixed number x

$$lim_{n\to\infty} P(\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma}) = \Phi(x),$$

where Φ denotes the c.d.f. of the standard normal distribution.

Central Limit Theorem

- $S_n = \sum_{i=1}^{n} X_i$, mean $n\mu$, variance $n\sigma^2$.
- $\bar{X}_n = \frac{S_n}{n}$, mean μ variance $\frac{\sigma^2}{n}$.
- $\frac{S_n}{\sqrt{n}}$, mean $\mu\sqrt{n}$, variance σ^2.
- $Zn = \frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$, mean 0, variance 1.

Example

- You are doing a poll on "ratio of people who believe it was a proper penalty".
- True ratio: p, estimate $\bar{X}_n$
- No guarantee for finding exactly p, so

$$P(|\bar{X}_n - p| \geq 0.01) \leq 0.05$$

- Apply Chebysev inequality with $t = 0.01$: n = 50,000.
- Apply CLT: ?

The Delta Method

Sometimes we are interested in the asymptotic behavior of a function of the sample mean.

Theorem (The Delta Method)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables from a distribution with mean μ and finite variance σ^2. Let α be a function with continuous derivative such that $\alpha(\mu)' \neq 0$. Then the asymptotic distribution of

$$\frac{n^{1/2}}{\sigma\alpha'(\mu)}[\alpha(\bar{X}_n) - \alpha(\mu)]$$

is the standard normal distribution.

Practice Exercises

6.2	2, 6, 8
6.3	3, 10, 13