Lecture Summary

5.5 The Negative Binomial Distributions

5.6 The Normal Distributions

5.7 ONLY: The Exponential Distributions

Negative Binomial distributions

Definition (Negative Binomial distribution)

A random variable X has the Negative Binomial distribution with parameters r and p if it has the pf

$$f(x|p,r) = \begin{cases} \binom{r+x-1}{x} p^r (1-p)^x & x = 0, 1, 2, \ldots, \\ 0 & otherwise. \end{cases}$$

where $0 < p < 1$ and r is a positive integer.

Say we have an infinite sequence of Bernoulli trials with parameter p, and X = number of "failures" before the r th "success". Then $X \sim NegBinomial(r, p)$.

- $E(X) = \frac{r(1-p)}{p}$
- $Var(X) = \frac{r(1-p)}{p^2}$
- MGF:

$$\psi(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$$

Geometric distributions

Definition (Geometric Distribution)

A r.v. X has the Geometric distribution with parameter p a if the probability function (pf) of X is

$$f(x|p) = \begin{cases} f(x|p) = p(1-p)^x & x = 0, 1, \ldots n \\ 0 & otherwise \end{cases}$$

- An experiment with two outcomes: "success", "failure", X = number of failures before the first success.
- Parameter space $p \in [0, 1]$.
- $E(X) = \frac{1-p}{p}$
- $Var(X) = \frac{1-p}{p^2}$.
- MGF:

$$\psi(t) = (\frac{p}{1-(1-p)e^t})$$

Properties of Geometric distributions

Theorem (Sum of Geometric is Negative Binomial)

If $X_1, ..., X_r$ are i.i.d. and each $X_i \sim Geometric(p)$ then $X = X_1 + \cdots + X_r \sim NegBinomial(r, p)$.

Theorem (Geometric distributions are memoryless:)

Let X have the geometric distribution with parameter p, and let $k \geq 0$. Then for every integer $t \geq 0$,

$$P(X = k + t | X \geq k) = P(X = t).$$

The Exponential Distributions

Definition (Exponential Distributions)

Let $\beta > 0$. A random variable X follows the *exponential distribution* with parameter β if it has a continuous distribution with pf:

$$f(x|\beta) = \begin{cases} \beta e^{-\beta x} & x > 0, \\ 0 & otherwise. \end{cases}$$

where $\beta > 0$

- $E(X) = \frac{1}{\beta}$
- $Var(X) = \frac{1}{\beta^2}$
- MGF:

$$\psi(t) = \frac{\beta}{\beta - t} \text{ for } t < \beta$$

Properties of the Exponential Distributions

Theorem (Exponential distributions are memoryless)

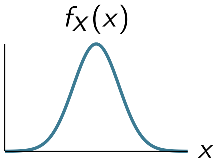
Let X have the exponential distribution with parameter β, and let $t > 0$. Then for every number $h > 0$,

$$P(X \geq t + h | X \geq t) = P(X \geq h)$$

Theorem (Minimum of exponentials is exponential)

Suppose $X_1, X_2, \ldots, X_n$ each follow an exponential distribution with parameter β. Then the distribution of $Y = min\{X_1, \ldots, X_n\}$ will be the exponential distribution with parameter $n\beta$.

The Normal Distribution



Standard normal

$$\mathcal{N}(0,1) : f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

.

Normal with mean μ and variance σ^2

$$\mathcal{N}(0,1) : f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

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Computing Probabilities for Normal r.v.s

- The cdf for a normal distribution cannot be expressed in closed form and is evaluated using numerical approximations.
- $\Phi(x)$ is the cdf of the standard normal, and it is tabulated in the back of most statistical books. Many calculators and programs such as R, Matlab, Excel etc. can calculate $\Phi(x)$
- $\Phi(-x) = 1 - \Phi(x)$
- $\Phi^{-1}(p) = -\Phi^{-1}(1 - p)$

Theorem

Linear transformation of a normal is still normal] If $X \sim N(\mu, \sigma^2)$ *and* $Y = aX + b$ *then* $Y \sim N(a\mu + b, a^2\sigma^2)$

- Let F be the cdf of X, where $X \sim N(\mu, \sigma^2)$.
- Then $F(x) = \Phi(\frac{x-\mu}{\sigma})$
- $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$

Linear Combinations of Independent Normals

Theorem (Linear Combinations of Independent Normals is a Normal.)

Let $X_1, \ldots, X_k$ be independent random variables and $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, k$. Then $X_1 + \cdots + X_k \sim N(\mu_1 + \cdots + \mu_k, \sigma_1^2 + \cdots + \sigma_k^2)$. Also, if $a_1, \ldots, a_k, b$ are constants where at least one a_i is not zero, then $a_1 X_1 + \cdots + a_k X_k + b \sim N(\sum_{i=1}^k a_i \mu_i + b, \sum_{i=1}^k a_i^2 \sigma_i^2)$.

- Assume $X_1, \ldots X_n$ are a random sample from $N(\mu, \sigma^2)$.
- What is the distribution of the sample mean, $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$?

Practice Exercises

5.2	6, 9,10, 13
5.4	4,5 , 9, 10
5.6	3, 4, 10, 14
5.7	6, 10
5.11	11,12