Lecture Summary

Moments and Central Moments

Definition (Moments and Central Moments)

Let X be a random variable and k be a positive integer.
The expectation $E(X^k)$ is the k-th moment of X. The expectation $E[(X - E(X))^k]$ is the k-th central moment of X.

- The first moment is the mean: $\mu = E(X^1)$.
- The first central moment is zero:
  $E[(X - E(X))^1] = E(X - \mu) = E(X) - E(X) = 0$
- The second central moment is the variance:
  $E[(X - E(X))^2] = Var(X)$

Moment Generating Functions

Definition (Moment Generating Functions)

Let X be a random variable. The function

$$\psi(t) = E(e^{tX}), t \in R$$

is called the moment generating function (m.g.f.) of X.

Theorem

Let X be a random variables whose m.g.f. $\psi(t)$ is finite for t in an open interval around zero. Then the $n-th$ moment of X is finite, for $n = 1, 2, \dots,$and

$$E(X^n) = \frac{d^n\psi(t)}{dt^n}|_{t=0}$$

Properties of Moment Generating Functions

- $\psi(aX + bt) = e^{bt}\psi_X(at)$.
- Let $Y = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are independent random variables with m.g.f $\psi_i(t)$ for $i = 1, \ldots, n$. Then

$$\psi_Y(t) = \prod_{i=1}^{n} X_i$$

- Let X and Y be two random variables with m.g.f.'s $\psi_X(t)$ and $\psi_Y(t)$. If the m.g.f.'s are finite and $\psi_X(t) = \psi_Y(t)$ for all values of t in an open interval around zero, then X and Y have the same distribution.

Finding the p.d.f's for sums of random variables

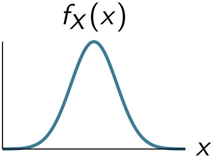
- Let $Y = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are independent random variables with m.g.f $\psi_i(t)$ for $i = 1, \ldots, n$. Then

$$\psi_Y(t) = \prod_{i=1}^{n} \psi_i(t)$$

Theorem

If X_1 and X_2 are independent random variables, and if X_i has the binomial distribution with parameters n_i and p $(i = 1, 2)$, then $X_1 + X_2$ has the binomial distribution with parameters $n_1 + n_2$ and p.

The Normal Distribution



Standard normal

$$\mathcal{N}(0,1) : f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

.

Normal with mean μ and variance σ^2

$$\mathcal{N}(0,1) : f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

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Computing Probabilities for Normal r.v.s

- The cdf for a normal distribution cannot be expressed in closed form and is evaluated using numerical approximations.
- $\Phi(x)$ is the cdf of the standard normal, and it is tabulated in the back of most statistical books. Many calculators and programs such as R, Matlab, Excel etc. can calculate $\Phi(x)$
- $\Phi(-x) = 1 - \Phi(x)$
- $\Phi^{-1}(p) = -\Phi^{-1}(1-p)$

Theorem

Linear transformation of a normal is still normal] If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ then $Y \sim N(a\mu + b, a^2\sigma^2)$

- Let F be the cdf of X, where $X \sim N(\mu, \sigma^2)$.
- Then $F(x) = \Phi(\frac{x-\mu}{\sigma})$
- $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$

Practice Exercises

4.4	1,2
5.6	3, 4, 10, 14