**Corollary**: Let $X_1, X_2, \dots$ be independent r.v's with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 < \infty$ $\forall i$. If $\sum_{n=1}^{\infty} \operatorname{Var}(X_n)$ converges, then $\sum_{n=1}^{\infty} X_n$ converges almost surely.

**Recall**: $\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n > m \geq N$ $|a_m + \dots + a_n| < \varepsilon$

**Pf** We have that

$$\mathbb{P}\Big(\sum_{n=1}^{\infty} X_n \text{ diverges}\Big) = \mathbb{P}\Big(\bigcup_{k \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \Big\{\sup_{n>m\geq N} |X_m + \dots + X_n| \geq \tfrac{1}{k}\Big\}\Big)$$

$$\leq \sum_{k=1}^{\infty} \mathbb{P}\Big(\bigcap_{N \in \mathbb{N}} \underbrace{\Big\{\sup_{n>m\geq N} |X_m + \dots + X_n| \geq \tfrac{1}{k}\Big\}}_{\mathcal{E}_N}\Big).$$

**Note**: $\mathcal{E}_N \supseteq \mathcal{E}_{N+1} \dots$

Now for $N \in \mathbb{N}$

$$\mathbb{P}(\mathcal{E}_N) = \mathbb{P}\Big(\sup_{n>m\geq N} |X_m + \dots + X_n| \geq \tfrac{1}{k}\Big) \quad \mathbb{P}\Big(\bigcup_{n>N} \max_{N \leq m < n} |X_m + \dots + X_n| \geq \tfrac{1}{k}\Big)$$

$$= \lim_{n\to\infty} \mathbb{P}\Big(\max_{N \leq m < n} |X_m + \dots + X_n| \geq \tfrac{1}{k}\Big)$$

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$$\leq \lim_{n\to\infty} \frac{1}{1/k^2} \mathbb{E}(X_N + \dots + X_n)^2 = \lim_{n\to\infty} k^2 \sum_{j=N}^{n} \operatorname{Var}(X_j)$$

$X_j$ independent, $\mathbb{E}X_j = 0$

Now since $\sum_{j=1}^{\infty} \operatorname{Var}(X_j) < \infty$ we have $\sum_{j=N}^{\infty} \operatorname{Var}(X_j) \to 0$ as $N \to \infty$.

In detail: fix $\varepsilon > 0$. For any $k \in \mathbb{N}$ $\exists N_k \in \mathbb{N}$ s.t. $\sum_{j=N_k}^{\infty} \operatorname{Var}(X_j) < \frac{\varepsilon}{k^4}$

Then

$$\mathbb{P}\Big(\sum_{n=1}^{\infty} X_n \text{ diverges}\Big) \leq \sum_{k=1}^{\infty} \lim_{N\to\infty} \lim_{n\to\infty} k^2 \sum_{j=N}^{n} \operatorname{Var}(X_j)$$

$$= \sum_{k=1}^{\infty} k^2 \underbrace{\lim_{N\to\infty} \sum_{j=N}^{\infty} \operatorname{Var}(X_j)}_{< \frac{\varepsilon}{k^4}} < \varepsilon \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$$

and since this holds for an arbitrary $\varepsilon > 0$ we get $\mathbb{P}\Big(\sum_{n=1}^{\infty} X_n \text{ diverges}\Big) = 0$