

# Lecture Summary

3.1 Discrete Random Variables

3.4 Bivariate Distributions

3.5 Marginal Distributions

3.6 Conditional Distributions

4.1 Expectations

Some slides from MIT open courseware

## Binomial Distribution

Now imagine you toss the fair coin 5 times. Some possible outcomes are 00000, 1000, 01100 etc. Let  $X$  be the number of successes (heads) after the 5 times. Then  $X$  follows a Binomial distribution with parameters (5, 0.5).

### Definition

The binomial distribution with parameters  $n$  and  $p$  is the discrete probability distribution of the number of successes in a sequence of  $n$  independent experiments, each with a binary outcome: success (with probability  $p$ ) or failure (with probability  $q = 1 - p$ ). The pmf of the binomial distribution is

$$P_X(x) = \binom{n}{x} p^x (1 - p)^{1-x} \text{ for } x = 0, \dots, n,$$

## Geometric distribution

Now imagine you toss the fair coin until you get heads. times. Some possible outcomes are 1, 01, 0001, 0...1, etc. Let  $X$  be the trial of the first success (heads). Then  $X$  follows a Geometric distribution parameter 0.5.

### Definition

## Joint Probability Mass Function

3	1/20	2/20	2/20	
2	2/20	3/20	1/20	2/20
1		2/20	3/20	
0	1/20			
	0	1	2	3

$$P_{X,Y}(x,y) = P(X = x, Y = y) = P(X = x \& Y = y)$$

$$\sum_{\text{all}(x,y)} P_{X,Y}(x,y) = 1 \text{ (still a probability mass function)}$$

$$\text{Marginal Probability: } P_X(x) = \sum_y P_{X,Y}(x,y) \text{ (sum over all possible } y \text{)}$$

# Joint Probability mass function

## Definition

Let  $X$  and  $Y$  be random variables. If there are at most countable possible outcomes  $(x, y)$  for the pair  $(X, Y)$ , we say that  $X$  and  $Y$  have a discrete joint distribution. The joint probability function (joint pf) is

$$P(X = x \text{ and } Y = y) := P(X = x, Y = y) \forall (x, y) \in R$$

As for the univariate case we have

- ▶  $P(x, y) \geq 0$
- ▶  $\sum_{(x,y)} P_{X,Y}(x, y) = 1$
  
- ▶ Two variables: Bivariate Distribution
- ▶ More than two variables: Multivariate Distribution

# Marginal Probability Mass Functions

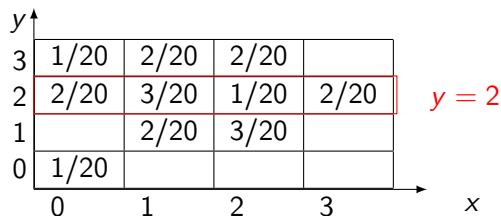
## Theorem

Let  $(X, Y)$  be a discrete random vector with joint probability mass function  $P_{X,Y}(x, y)$ , then the marginal pmfs of  $X$  and  $Y$  are given by

$$P_X(x) = \sum_{y \in R} P_{X,Y}(x, y)$$

$$P_Y(y) = \sum_{x \in R} P_{X,Y}(x, y)$$

## Conditional Probability Mass Functions



$$\text{Conditional Probability: } P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$\text{e.g., } P_{X|Y}(x|y=2) = \{2/9, 4/9, 1/9, 2/9\}$$

$$\sum_x P_{X|Y}(x|y) = 1 \text{ (still a probability mass function)}$$

# Independence of Random Variables

Independence for random variables is defined in the same way as for events

## Definition (Independent Random Variables)

Two random variables are independent if for every two sets  $A$  and  $B$  in  $R$  the events  $\{s : X(s) \in A\}$  and  $\{s : Y(s) \in B\}$  are independent

## Theorem

*Two random variables  $X$  and  $Y$  with joint pmf  $P_{X,Y}(x,y)$  and marginal pmfs  $P_X(x)$  and  $P_Y(y)$  are independent if and only if*

$$P_{X,Y}(x,y) = P_X(x)P_Y(y) \text{ for all } (x,y) \in R^2$$



## Conditional Independence

3	1/20	2/20	2/20	
2	2/20	3/20	1/20	2/20
1		2/20	3/20	
0	1/20			
	0	1	2	3

- ▶ Assume you know that  $X \leq 2$  and  $Y \geq 4$
- ▶ Are  $X$  and  $Y$  independent in this new universe?

Here we conditioned on an event, but in general  $X$  and  $Y$  are independent given  $Z$  if

$$P_{X,Y|Z}(x,y|z) = P_{X|Z}(x|z)P_{Y|Z}(y|z) \quad \forall (x,y,z) \in R^3$$

## Exercise

A fair coin is tossed three times. Let

- ▶  $X$  = number of heads on the first toss
- ▶  $Y$  = total number of heads
- ▶ Find the joint distribution of  $X, Y$ .
- ▶ Find the marginal distributions of  $X$  and  $Y$ .
- ▶ Find the conditional distribution of  $Y|X$
- ▶ Are  $X$  and  $Y$  independent?

## Definition (Expectation)

The **expected value** or **mean** of  $X$  is defined to be

$$E(X) = \sum_x xP_X(x)$$

assuming that the sum is well-defined.

We can think of the expectation as the average of a very large number of independent draws from the distribution (IID draws).

## Examples

- ▶ Let  $X \sim \text{Bernoulli}(p)$ .  $E(X) = ?$
- ▶ Flip a fair coin twice. Let  $X$  be the number of heads.  $E(X) = ?$

# Practice Exercises

Section	Exercises
3.1	9, 10
3.4	2
3.5	2,5
3.6	2