Lecture Summary

3.1 Discrete Random Variables

3.4 Bivariate Distributions

3.5 Marginal Distributions

3.6 Conditional Distributions

4.1 Expectations

Some slides from MIT open courseware

Binomial Distribution

Now imagine you toss the fair coin 5 times. Some possible outcomes are 00000, 1000, 01100 etc. Let X be the number of successes (heads) after the 5 times. Then X follows a Binomial distribution with parameters (5, 0.5).

Definition

The binomial distribution with parameters n and p is the discrete probability distribution of the number of successes in a sequence of n independent experiments, each with a binary outcome: success (with probability p) or failure (with probability $q = 1 - p$). The pmf of the binomial distribution is

$$P_X(x) = \binom{n}{x} p^x (1-p)^{1-x} \text{ for } x = 0, \ldots, n,$$

Geometric distribution

Now imagine you toss the fair coin until you get heads. times. Some possible outcomes are 1, 01, 0001, $0 \ldots 1$, etc. Let X be the trial of the first success (heads). Then X follows a Geometric distribution parameter 0.5.

Definition

Joint Probability Mass Function

y \ x	0	1	2	3
3	1/20	2/20	2/20	
2	2/20	3/20	1/20	2/20
1		2/20	3/20	
0	1/20			

$$P_{X,Y}(x,y) = P(X = x, Y = y) = P(X = x \& Y = y)$$

$$\sum_{all(x,y)} P_{X,Y}(x,y) = 1 \text{ (still a probability mass function)}$$

$$\text{Marginal Probability:} P_X(x) = \sum_{y} P_{X,Y}(x,y) \text{ (sum over all possible y)}$$

Joint Probability mass function

Definition

Let X and Y be random variables. If there are at most countable possible outcomes (x, y) for the pair (X, Y), we say that X and Y have a discrete joint distribution. The joint probability function (joint pf) is

$$P(X = x \text{ and } Y = y) := P(X = x, Y = y) \forall (x, y) \in R$$

As for the univariate case we have

- $P(x, y) \geq 0$
- $\sum_{(x,y)} P_{X,Y}(x, y) = 1$

- Two variables: Bivariate Distribution
- More than two variables: Multivariate Distribution

Marginal Probability Mass Functions

Theorem

Let (X, Y) be a discrete random vector with joint probability mass function $P_{X,Y}(x, y)$, then the marginal pmfs of X and Y are given by

$$P_X(x) = \sum_{y \in R} P_{X,Y}(x, y)$$

$$P_Y(y) = \sum_{x \in R} P_{X,Y}(x, y)$$

Conditional Probability Mass Functions

y \ x	0	1	2	3
3	1/20	2/20	2/20	
2	2/20	3/20	1/20	2/20
1		2/20	3/20	
0	1/20			

$y = 2$

$$\text{Conditional Probability:} P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$\text{e.g., } P_{X|Y}(x|y=2) = \{2/9, 4/9, 1/9, 2/9\}$$

$$\sum_x P_{X|Y}(x|y) = 1 \text{ (still a probability mass function)}$$

Independence of Random Variables

Independence for random variables is defined in the same way as for events

Definition (Independent Random Variables)

Two random variables are independent if for every two sets A and B in R the events $\{s : X(s) \in A\}$ and $\{s : Y(s) \in B\}$ are independent

Theorem

Two random variables X and Y with joint pmf $P_{X,Y}(x, y)$ and marginal pmfs $P_X(x)$ and $P_Y(y)$ are independent if and only if

$$P_{X,Y}(x, y) = P_X(x)P_Y(y) \text{ for all} (x, y) \in R^2$$

Conditional Independence

y \ x	0	1	2	3
3	1/20	2/20	2/20	
2	2/20	3/20	1/20	2/20
1		2/20	3/20	
0	1/20			

- Assume you know that $X \leq 2$ and $Y \geq 4$
- Are X and Y independent in this new universe?

Here we conditioned on an event, but in general X and Y are independent given Z if

$$P_{X,Y|Z}(x,y|z) = P_{X|Z}(x|z)P_{Y|Z}(y|z) \quad \forall (x,y,z) \in R^3$$

Exercise

A fair coin is tossed three times. Let

- X = number of heads on the first toss
- Y = total number of heads

- Find the joint distribution of X, Y.
- Find the marginal distributions of X and Y.
- Find the conditional distribution of $Y|X$
- Are X and Y independent?

Definition (Expectation)

The **expected value** or **mean** of X is defined to be

$$E(X) = \sum_{x} x P_X(x)$$

assuming that the sum is well-defined.

We can think of the expectation as the average of a very large number of independent draws from the distribution (IID draws).

Examples

- Let $X \sim Bernoulli(p). E(X) = ?$
- Flip a fair coin twice. Let X be the number of heads. $E(X) = ?$

Practice Exercises

Section	Exercises
3.1	9, 10
3.4	2
3.5	2,5
3.6	2