Parametric Statistics-Recitation 10 (Solutions)

Exercise 1.

Let X have the exponential distribution with parameter β. Suppose that we wish to test the hypotheses $H_0 : \beta \geq 1$ versus $H_1 : \beta < 1$. Consider the test procedure δ that rejects H_0 if $X \geq 1$.

(a) Determine the power function of the test.

(b) Compute the size of the test.

Solution

(a) Let δ be the test that rejects H_0 when $X \geq 1$. The power function of δ is

$$\pi(\beta|\delta) = Pr(X \geq 1|\beta) = 1 - Pr(X < 1|\beta) = e^{-\beta}$$

for $\beta > 0$.

(b) The size of the test δ is $sup_{\beta \geq 1}\pi(\beta|\delta)$. We found that $\pi(\beta|\delta)$ is a decreasing function of β, hence the size of the test is $\pi(1|\delta) = e^{-1}$.

Exercise 2.

Suppose that $X_1, \ldots, X_n$ form a random sample from the uniform distribution on the interval $[0, \theta]$, and that the following hypotheses are to be tested:

$$\begin{aligned} H_0 : &\quad \theta \geq 2, \\ H_1 : &\quad \theta < 2. \end{aligned}$$

Let $Y_n = \max\{X_1, \ldots, X_n\}$, and consider a test procedure such that the critical region contains all the outcomes for which $Y_n \leq 1.5$.

(a) Determine the power function of the test.

(b) Compute the size of the test.

Solution

(a) We know that if $0 < y < \theta$, then $Pr(Y_n \leq y) = \left(\frac{y}{\theta}\right)^n$. Also, if $y \geq \theta$, then $Pr(Y_n \leq y) = 1$.
Therefore, if $\theta \leq 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = 1$. If $\theta > 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = \left(\frac{1.5}{\theta}\right)^n$.

(b) The size of the test is

$$a = sup_{\theta \geq 2}\pi(\theta) = sup_{\theta \geq 2}\left(\frac{1.5}{\theta}\right)^n = \left(\frac{1.5}{2}\right)^n = \left(\frac{3}{4}\right)^n$$

Exercise 3.

Suppose that the proportion p of defective items in a large population of items is unknown, and that it is desired to test the following hypotheses:

$$\begin{aligned} H_0 : &\quad p = 0.2, \\ H_1 : &\quad p \neq 0.2. \end{aligned}$$

Suppose also that a random sample of 20 items is drawn from the population. Let Y denote the number of defective items in the sample, and consider a test procedure δ such that the critical region contains all the outcomes for which either $Y \geq 7$ or $Y \leq 1$.

(a) Determine the value of the power function $\pi(p \mid \delta)$ at the points $p = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9,$ and 1 ; sketch the power function.

(b) Determine the size of the test.

Solution

(a) For any given value of p, $\pi(p) = Pr(Y \geq 7) + Pr(Y \leq 1)$, where Y has a binomial distribution with parameters $n = 20$ and p. For $p = 0$, $\pi(0) = 1$. For$p = 0.1$, it is found from the table of the binomial distribution that $\pi(0.1) = 0.3941$. We found the remaining values from the table with similar way.

(b) Since H_0 is a simple hypothesis, the size a of the test is just the value of the power function at the point specified by H_0. Thus, $a = \pi(0.2) = 0.1558$.

Exercise 4.

Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and known variance 1 . Suppose also that μ_0 is a certain specified number, and that the following hypotheses are to be tested:

$$\begin{aligned} H_0 : &\quad \mu = \mu_0, \\ H_1 : &\quad \mu \neq \mu_0. \end{aligned}$$

Finally, suppose that the sample size n is 25 , and consider a test procedure such that H_0 is to be rejected if $\left|\bar{X}_n - \mu_0\right| \geq$ c. Determine the value of c such that the size of the test will be 0.05 .

Solution

The null hypothesis H_0 is simple. Therefore, the size a of the test is $a = Pr(Rejecting H_0 | \mu = \mu_0)$. When $\mu = \mu_0$, the random variable $Z = n^{1/2}(\bar{X}_n - \mu_0)$ will have the standard normal distribution. Hence, since $n = 25$,

$$a = Pr\left(|\bar{X}_n - \mu_0| \geq c\right) = Pr(|Z| \geq 5c)$$

Thus, $a = 0.05$ if and only if $5c = 1.96$ or $c = 0.392$.

Exercise 5.

Suppose that a certain drug A was administered to eight patients selected at random, and after a fixed time period. the concentration of the drug in certain body cells of each patient was measured in appropriate units Suppose that these concentrations for the eight patients were found to be as follows:

$$1.23, 1.42, 1.41, 1.62, 1.55, 1.51, 1.60, \text{ and } 1.76.$$

Suppose also that a second drug B was administered to six different patients selected at random, and when the concentration of drug B was measured in a similar way for these six patients, the results were as follows:

$$1.76, 1.41, 1.87, 1.49, 1.67, \text{ and } 1.81.$$

Assuming that all the observations have a normal distribution with a common unknown variance, test the following hypotheses at the level of significance 0.10 : The null hypothesis is that the mean concentration of drug A among all patients is at least as large as the mean concentration of drug B. The alternative hypothesis is that the mean concentration of drug B is larger than that of drug A.

Solution

In this exercise $m = 8$, $n = 6$, $\bar{X}_m = 1.51125$, $\bar{Y}_n = 1.6683$, $S_X^2 = 0.18075$ and $S_Y^2 = 0.16768$. When $\mu_1 = \mu_2$ the statistic U will have the t-distribution with 12 degrees of freedom, where

$$U = \frac{\sqrt{m+n-2}(\bar{X}_m - \bar{Y}_n)}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}}$$

We reject H_0 is $U < c$. Since the level of significance is 0.1 is is found from the table that $c = -1.356$. The calculated value of U is -1.692. Therefore, H_0 is rejected.

Exercise 6.

Bottles of a popular cola drink are supposed to contain 300 milliliters (ml) of cola. There is some variation from bottle to bottle because the filling machinery is not perfectly precise. The distribution of contents is normal, with known variance $\sigma^2 = 9$. A student measures the contents of 6 bottles. The results are $299.4, 297.7, 301.0, 298.9, 300.2, 297.0$, with sample mean equal to 299.03. Is this convincing evidence that the mean contents μ of cola bottles is different than the advertised 300ml, at level $\alpha = 0.05$?

Solution

The hypotheses that we will test are

$$\begin{aligned} H_0 : &\quad \mu = 300, \\ H_1 : &\quad \mu \neq 300. \end{aligned}$$

We reject H_0 if $p - value < a = 0.05$ that is if

$$Pr(|Z| \geq c) < 0.05$$

$$Z \leq -1.96 \text{ or } Z \geq 1.96$$

For our data we have $Z = -0.7893$ so there is no convincing evidence to reject the null hypothesis.

Exercise 7.

Suppose that a random variable X has the F distribution with three and eight degrees of freedom. Determine the value of c such that $\Pr(X > c) = 0.975$.

Solution

Let $Y = \frac{1}{X}$. Then Y has the F distribution with 8 and 3 degrees of freedom. Also

$$Pr(X > c) = Pr(Y < \frac{1}{c}) = 0.975$$

From the tables of F distribution we found that $Pr(Y < 14.54) = 0.975$. Therefore, $c = 0.069$.

Exercise 8.

Consider two different normal distributions for which both the means μ_1 and μ_2 and the variances σ_1^2 and σ_2^2 are unknown, and suppose that it is desired to test the following hypotheses:

$$\begin{aligned} H_0 : &\quad \sigma_1^2 \leq \sigma_2^2, \\ H_1 : &\quad \sigma_1^2 > \sigma_2^2. \end{aligned}$$

Suppose further that a random sample consisting of 16 observations for the first normal distribution yields the values $\sum_{i=1}^{16} X_i = 84$ and $\sum_{i=1}^{16} X_i^2 = 563$, and an independent random sample consisting of 10 observations from the second normal distribution yields the values $\sum_{i=1}^{10} Y_i = 18$ and $\sum_{i=1}^{10} Y_i^2 = 72$.

(a) What are the M.L.E.'s of σ_1^2 and σ_2^2 ?

(b) If an F test is carried out at the level of significance 0.05 , is the hypothesis H_0 rejected or not?

Solution

(a) Here, $\bar{X_m} = \frac{84}{16} = 5.25$ and $\bar{Y_n} = \frac{18}{10} = 1.8$. Therefore,

$$S_1^2 = \sum_{i=1}^{16} X_i^2 - 16\bar{X_m}^2 = 122 \text{ and } S_2^2 = \sum_{i=1}^{10} Y_i^2 - 10\bar{Y_n}^2 = 39.6$$

It follows that

$$\hat{\sigma_1^2} = \frac{1}{16} S_1^2 = 7.625 \text{ and } \hat{\sigma_2^2} = \frac{1}{10} S_2^2 = 3.96$$

(b) If $\sigma_1^2 = \sigma_2^2$, the following statistic V will have the F distribution with 15 and 9 degrees of freedom

$$V = \frac{\frac{S_1^2}{15}}{\frac{S_2^2}{9}}$$

For the level of significance 0.05, H_0 should be rejected if $V > 3.01$. It is found that $V = 1.848$. Therefore we do not reject H_0.