Lecture Summary

- Joint/Conditional/Marginal PMFs.
- Joint/Conditional/Marginal PDFs.
- Covariance.
- Conditional Expectations.
- Properties of expecations

Material can be found in Chapters 3 (3.4-3.9) and 4 (4.1-4.3, 4.6-4.7) of DeGroot and Schervish.
Many slides from MIT open courseware

Joint PMFs

y \ x	0	1	2	3
3	1/20	2/20	2/20	
2	2/20	3/20	1/20	2/20
1		2/20	3/20	
0	1/20			

$$P_{X,Y}(x,y) = P(X = x, Y = y) = P(X = x \& Y = y)$$

$$\sum_x \sum y P_{X,Y}(x,y) = 1 \text{ (still a probability mass function)}$$

$$\text{Marginal Probability:} P_X(x) = \sum_y P_{X,Y}(x,y) \text{ (sum over all possible y)}$$

Conditional PMFs

y \ x	0	1	2	3
3	1/20	2/20	2/20	
2	2/20	3/20	1/20	2/20
1		2/20	3/20	
0	1/20			

$y = 2$

$$\text{Conditional Probability:} P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$\text{e.g., } P_{X|Y}(x|y=2) = \{2/9, 4/9, 1/9, 2/9\}$$

$$\sum_x P_{X|Y}(x|y) = 1 \text{ (still a probability mass function)}$$

Joint/Conditional/Marginal PDFs

$$P(X, Y \in S) = \int \int_S f_{X,Y} dxdy$$

$$f_{X,Y}(x, y) \approx P(x \leq X \leq x + \delta, y \leq Y \leq y + \delta)\delta^2$$

$$\int_x \int y f_{X,Y}(x, y) = 1 \text{ (still a probability density function)}$$

$$\text{Marginal Probability:} f_X(x) = \int_y f_{X,Y}(x, y) \text{ (integrate over all possible y)}$$

Joint/Conditional/Marginal PDFs

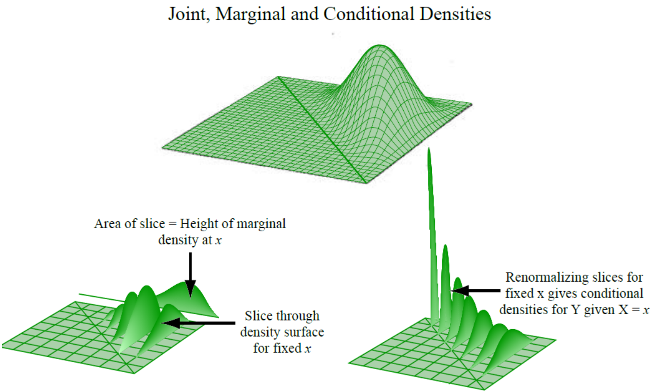


Image by MIT OpenCourseWare, adapted from *Probability*, by J. Pittman, 1999.

Expectations

LOTUS for functions of multiple r.v.s:

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) P_{X,Y}(x,y), \text{ discrete}$$

$$E[g(X,Y)] = \int_x \int_y g(x,y) f_{X,Y}(x,y) dxdy, \text{ continuous}$$

Conditional Expectation

$$E(X|y) = \sum_x x P_{X|y} \text{ (for a given value } y \text{ of } Y)$$

$$E(X|Y) = \sum_x x P_{X|Y} \text{(for every value } y \text{ of } Y)$$

Independent random variables

Independent Discrete Random Variables

$$P_{X,Y}(x, y) = P_X(x)P_Y(y) \text{ for every pair } (x,\ y)$$

Independent Continuous Random Variables

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \text{ for every pair } (x,\ y)$$

We can extend this to multiple random variables.

Linearity of Variances for Independent Random Variables

Linearity of variances only holds for independent random variables.

Theorem

Let $X_1, \ldots, X_n$ be a set of independent random variables. Then

$$Var[X_1 + \cdots + X_n] = Var[X_1] + \cdots + Var[X_n]$$

Prove it for the case of two discrete variables.

Covariance

Covariance measures how much two r.vs. vary together (i.e., are larger than usual at the same time).

Definition (Covariance)

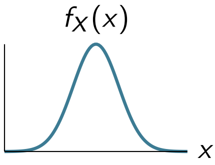
The **covariance** of two variables X and Y is defined to be

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$

If X, Y are independent, then $Cov(X, Y) = 0$ Correlation: Covariance without dimensions

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

The normal distribution



Standard normal

$$\mathcal{N}(0,1) : f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

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Normal with mean μ and variance σ^2

$$\mathcal{N}(0,1) : f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

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