Lecture Summary

- Expectation and variance of discrete random variables.
- Continuous Random Variables
- Cumulative Density Function
- Mixed Variables

Material can be found in Chapters 3 (3.1-3.3) and 4 (4.1-4.3) of Degroot and Schervish.

Definition (Expectation)

The **expected value** or **mean** or **first moment** of X is defined to be

$$E(X) = \sum_x x P_x(x)$$

assuming that the sum is well-defined.

- We can think of the expectation as the average of a very large number of independent draws from the distribution (IID draws).
- The fact that $E(X) = \sum_{i=1}^{n} X_i$ is actually a very important theorem we will discuss later.

Examples

- Let $X \sim Bernoulli(p). E(X) =?$
- Flip a fair coin twice. Let X be the number of heads. $E(X) =?$

Expectation of a function of a random variable

Sometimes we are interested in the expectation of a function of a random variable $Y = r(X)$. One way to find the expectation of this random variable:

- Find its pmf $P_Y(y)$
- Compute $\sum_y yP_Y(y)$

e.g. Assume X^2 is a discrete random variable with possible values $\{-3, -1, 0, 1, 3\}$ with probabilities $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$. Let $Y = X^2$. What is the expectation of Y?

Law Of The Unconscious Statistician

A simple way to compute the expectation of Y in the example above, or any function of random variables, is the law of unconscious statistician (LOTUS).

Theorem

Let $Y = r(X)$. Then

$$E(Y) = E(r(x)) = \sum_{\text{all } x} r(x)P_x(x)$$

if the mean exists.

Properties of Expectation

- $E(a) = a$
- $E(aX) = aE(X)$
- $E(aX + b) = aE(X) + b$
- But $E(g(X)) \neq g(E(X))$ in most cases!

Linearity of Expectation

Linearity of expectation is the property that the expected value of the sum of random variables is equal to the sum of their individual expected values, regardless of whether they are independent.

Theorem

Let $X_1, \ldots, X_n$ be a set of random variables. Then

$$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$$

Prove it for the case of two discrete variables.

Variance of a random variable

Sometimes we are also interested in quantifying how far from the mean

Definition (Variance)

The **variance** of X is defined to be

$$Var(X) = E[(X - E(X))^2]$$

assuming that the sum is well-defined.

Properties of Variance

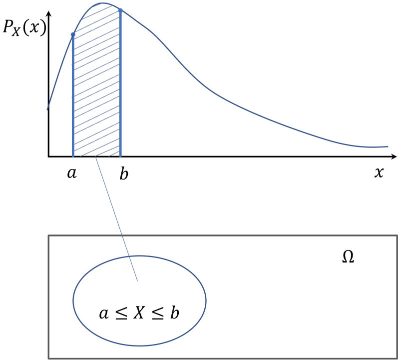
- $Var(a) = 0$
- $Var(aX) = a^2 Var(X)$
- $Var(aX + b) = a^2 Var(X)$

Examples

- Let $X \sim Bernoulli(p). Var(X) =?$
- Flip a fair coin twice. Let X be the number of heads. $Var(X) =?$

Continuous Random Variables

What if an r.v. takes values in a continuous range?



- $P(a \leq X \leq b) = \int_a^b f_X(x)dx$
- $P(x \leq X \leq x + \delta) = f_x(x)\delta$
- $\int_{-\infty}^{\infty} f_X(x) = 1$
- Density is not probability! We can integrate to compute probabilities.

Cumulative Distribution Function

Definition

The cumulative distribution function (CDF) is

$$F_X(x) = P(X \leq x) = \begin{cases} \int_{-\infty}^{x} f_X(t)dt, & \text{if } X \text{ is continuous} \\ \sum_{k \leq x} p_X(x) & \text{if } X \text{ is discrete} \end{cases}$$

- The CDF is non-decreasing as X increases.
- $lim_{x \to -\infty} F_X(x) = 0$, $lim_{x \to \infty} F_X(x) = 1$.
- The CDF is continuous from the right.
- $P(a < X \leq b) = F_X(b) - F_X(a)$.
- The CDF is well-defined for mixed variables.
- The pdf is the derivative of the CDF (where the derivative exists).