Chapter 9: Hypothesis Testing

Sections

- 9.5 The t Test
- 9.6 Comparing the Means of Two Normal Distributions

The *t*-Test

- The *t*-Test is a test for hypotheses concerning the mean parameter in the normal distribution when the variance is also unknown.
- The test is based on the *t* distribution

The setup for the next few slides:

- Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ and consider the hypotheses

$$H_0 : \mu \leq \mu_0 \qquad \text{vs.} \qquad H_1 : \mu > \mu_0 \tag{1}$$

The parameter space here is $-\infty < \mu < \infty$ and $\sigma^2 > 0$, i.e.

$$\Omega = (-\infty, \infty) \times (0, \infty)$$

And

$$\Omega_0 = (-\infty, \mu_0] \times (0, \infty) \quad \text{and} \quad \Omega_1 = (\mu_0, \infty) \times (0, \infty)$$

The one-sided t-Test

- The t test: a likelihood ratio test (see p. 583 - 585 in the book)
- Let

$$U = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{\sigma'} \quad \text{where} \quad \sigma' = \left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \right)^{1/2}$$

- If $\mu = \mu_0$ then U has the t_{n-1} distribution
- Tests based on U are called *t tests*

The one-sided t-Test

- Let T_m^{-1} be the quantile function of the t_m distribution
- The test δ that rejects H_0 in (1) if $U \geq T_{n-1}^{-1}(1-\alpha_0)$ has size α_0 (Theorem 9.5.1)
- To calculate the p-value:

Theorem 9.5.2: p-values for t Tests

Let u be the observed value of U.
The p-value for the hypothesis in (1) is $1 - T_{n-1}(u)$.

Example

Example: Acid Concentration in Cheese (Example 8.5.4)

- Have a random sample of $n = 10$ lactic acid measurements from cheese, assumed to be from a normal distribution with unknown mean and variance.
- Observed: $\overline{x}_n = 1.379$ and $\sigma' = 0.3277$
- Perform the level $\alpha_0 = 0.05$ t-test of the hypotheses

$$H_0 : \mu \leq 1.2 \quad \text{vs} \quad H_1 : \mu > 1.2$$

- Compute the p-value

The complete power function

- Need the power function to decide the sample size n
- The power function $\pi(\mu, \sigma^2|\delta)$ is a non-central t_m distributions

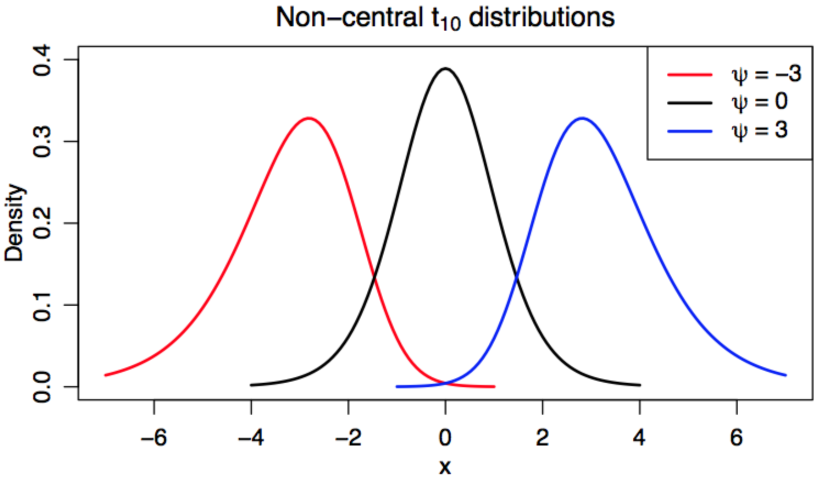
Def: Non-central t_m distributions

Let $W \sim N(\psi, 1)$ and $Y \sim \chi^2_m$ be independent. The distribution of

$$X = \frac{W}{(Y/m)^{1/2}}$$

is called the *non-central t distribution with m degrees of freedom and non-centrality parameter* ψ

Non-central t_m distribution



The complete power function

For the one-sided t-test

Theorem 9.5.3

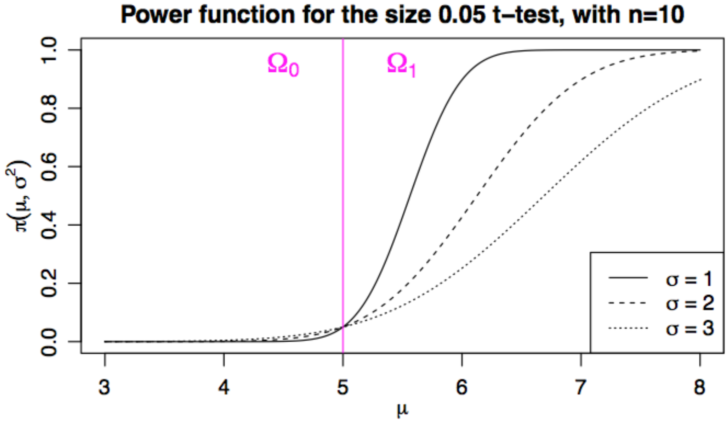
U has the non-central t_{n-1} distribution with non-centrality parameter $\psi = \sqrt{n}(\mu - \mu_0)/\sigma$.

The power function of the t-test that rejects H_0 in (1) if $U \geq T_{n-1}^{-1}(1 - \alpha_0) = c_1$ is

$$\pi(\mu, \sigma^2 | \delta) = 1 - T_{n-1}(c_1 | \psi)$$

Power function for the one-sided t-test

Example: $n = 10$, $\mu_0 = 5$, $\alpha_0 = 0.05$



Note that the power function is a function of both σ^2 and μ

The other one-sided t-Test

- Now consider the hypothesis

$$H_0 : \mu \geq \mu_0 \qquad \text{vs.} \qquad H_1 : \mu < \mu_0 \tag{2}$$

- The test δ that rejects H_0 if $U \leq T_{n-1}^{-1}(\alpha_0)$ has size α_0 (Corollary 9.5.1)

Theorem 9.5.2: p-values for t Tests

Let u be the observed value of U. The p-value for the hypothesis in (2) is $T_{n-1}(u)$.

Theorem 9.5.3

U has the non-central t_{n-1} distribution with non-centrality parameter $\psi = \sqrt{n}(\mu - \mu_0)/\sigma$. The power function of the t-test that rejects H_0 in (2) if $U \leq T_{n-1}^{-1}(\alpha_0) = c_2$ is

$$\pi(\mu, \sigma^2|\delta) = T_{n-1}(c_2|\psi)$$

Two-sided t-test

- Consider now the test with a two-sided alternative hypothesis:

$$H_0 : \mu = \mu_0 \qquad \text{vs.} \qquad H_1 : \mu \neq \mu_0 \tag{3}$$

- Size α_0 test δ: rejects H_0 iff $|U| \geq T_{n-1}^{-1}(1 - \alpha_0/2) = c$
- If u is the observed value of U then the p-value is $2(1 - T_{n-1}(|u|))$
- The power function is

$$\pi(\mu, \sigma^2|\delta) = T_{n-1}(-c|\psi) + 1 - T_{n-1}(c|\psi)$$

Notes on one sample t tests

- Paired t tests are conducted in the same way
- For large n, the distribution of the test statistic under H_0 is close to the standard normal, i.e., the corresponding test is close to a Z test

The two-sample t-test

Comparing the means of two populations

- $X_1, \ldots, X_m$ i.i.d. $N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_n$ i.i.d. $N(\mu_2, \sigma^2)$
- The variance is the same for both samples, but unknown

We are interested in testing one of these hypotheses:

a) $H_0 : \mu_1 \leq \mu_2$ vs. $H_1 : \mu_1 > \mu_2$

b) $H_0 : \mu_1 \geq \mu_2$ vs. $H_1 : \mu_1 < \mu_2$

c) $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$

Two-sample t statistic

Let $$\overline{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i \quad \text{and} \quad \overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$S_X^2 = \sum_{i=1}^{m}(X_i - \overline{X}_m)^2 \quad \text{and} \quad S_Y^2 = \sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2$$

$$U = \frac{\sqrt{m+n-2}\,(\overline{X}_m - \overline{Y}_n)}{\left(\frac{1}{m} + \frac{1}{n}\right)^{1/2}\left(S_X^2 + S_Y^2\right)^{1/2}}$$

- Theorem 9.6.1: If $\mu_1 = \mu_2$ then $U \sim t_{m+n-2}$
- Theorem 9.6.4: For any μ_1 and μ_2, U has the non-central t_{m+n-2} distribution with non-centrality parameter

$$\psi = \frac{\mu_1 - \mu_2}{\sigma\,(1/m + 1/n)^{1/2}}$$

Two-sample t test – summary

Proofs similar to the regular t-test

a) $H_0 : \mu_1 \leq \mu_2$ vs. $H_1 : \mu_1 > \mu_2$
   - Level α_0 test: Reject H_0 iff $U \geq T^{-1}_{m+n-2}(1 - \alpha_0)$
   - p-value: $1 - T_{m+n-2}(u)$

b) $H_0 : \mu_1 \geq \mu_2$ vs. $H_1 : \mu_1 < \mu_2$
   - Level α_0 test: Reject H_0 iff $U \leq T^{-1}_{m+n-2}(\alpha_0)$
   - p-value: $T_{m+n-2}(u)$

c) $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$
   - Level α_0 test: Reject H_0 iff $|U| \geq T^{-1}_{m+n-2}(1 - \alpha_0/2)$
   - p-value: $2(1 - T_{m+n-2}(|u|))$

- Power function is now a function of 3 parameters: $\pi(\mu_1, \mu_2, \sigma^2|\delta)$
- The two-sample *t*-test is a likelihood ratio test (see p. 592)
- Important difference: **Paired** *t* test vs. two sample *t* test
- Two-sample *t* test with unequal variances
  - Proposed test-statistics do not have known distribution, but approximations have been obtained
  - Approach 1: The Welch statistic

$$V = \frac{\overline{X}_m - \overline{Y}_n}{\left(\frac{S_X^2}{m(m-1)} + \frac{S_Y^2}{n(n-1)}\right)^{1/2}}$$

    can be approximated by a *t* distribution
  - Approach 2: The distribution of the likelihood ratio statistic can be approximated by the χ_1^2 distribution if the sample size is large enough