Chapter 8: Sampling distributions of estimators

(source: https://www2.stat.duke.edu/courses/Fall18/sta611.01/Lecture/Lecture12.pdf)

- 8.1 Sampling distribution of a statistic
- 8.2 The Chi-square distributions
- 8.3 Joint Distribution of the sample mean and sample variance
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- 8.4 The *t* distributions
  - Skip: derivation of the pdf, p. 483 - 484
- 8.5 Confidence intervals

Sampling distribution

- Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is a random sample from $f(x|\theta)$
- A *Sampling distribution*: the distribution of a statistic (given θ)
- Can use the sampling distributions to compare different estimators and to determine the sample size we need
- Used to get confidence intervals and to do hypothesis testing
- Leads to definitions of new distributions, e.g. χ^2_m and t_m distributions

Sampling distribution

Example:

- Suppose we want to use a statistic $T = r(X_1, \dots, X_n)$ as an estimate of a parameter θ
- To be able to calculate things like

$$P(|T - \theta| < 0.05)$$

  we need to know the distribution of T

- Let $X_1, \ldots, X_n$ be a random sample
- The *sample mean* and the *sample variance* are defined as

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

A familiar example:

- Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \sigma^2)$ where σ^2 is known
- The sample mean $T = \overline{X}_n$ is a statistic and $N(\theta, \sigma^2/n)$ is the *sampling distribution* of T

Theorem 8.3.1

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Then $\overline{X}_n$ and S_n are independent random variables and

$$\overline{X}_n \sim N(\mu, \sigma^2/n) \quad \text{and}$$
$$\frac{n}{\sigma^2} S_n = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \sim \chi^2_{n-1}$$

The Chi-square distributions

Def: Chi-square distributions

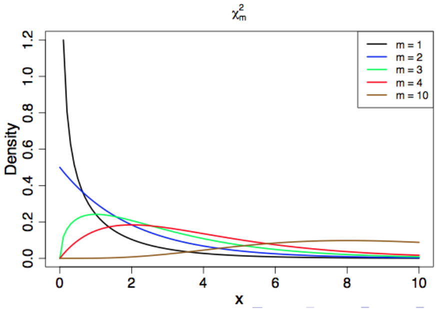
The χ^2_m *distribution with m degrees of freedom (df)* is the $\text{Gamma}(\alpha = m/2, \beta = 1/2)$. The pdf is

$$f(x|m) = \frac{1}{2^{m/2}\Gamma(m/2)} x^{m/2-1} e^{-x/2}$$

If $X \sim \chi^2_m$ then

- $E(X) = m$ and
- $\text{Var}(X) = 2m$ and
- $\psi(t) = \left(\frac{1}{1-2t}\right)^{m/2}$

The χ^2_m distribution is tabulated at the end of the book



Properties of the χ^2_m distributions

And connections to the normal distributions

Theorem 8.2.1: A sum of chi-squares is a chi-square

Let $X_1, \ldots, X_n$ be independent random variables and $X_i \sim \chi^2_{m_i}$. Then

$$X_1 + \cdots + X_n \ \sim \ \chi^2_m$$

where $m = m_1 + \cdots + m_n$

- Follows directly from the fact that a sum of $\mathrm{Gamma}(\alpha_i, \beta)$ random variables (same β) is a $\mathrm{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$

Theorem 8.2.2: Square of a standard normal is a chi-square

If $X \sim N(0, 1)$ then $X^2 \sim \chi^2_1$

Properties of the χ^2_m distributions

And connections to the normal distributions

Corollary 8.2.1

If the random variables $X_1, \dots, X_n$ i.i.d. $N(0,1)$ then

$$X_1^2 + \cdots + X_n^2 \sim \chi^2_n$$

When do we use the χ^2_m distribution?

- If $X_1, \dots, X_n$ are i.i.d. $N(\mu, \sigma^2)$ where μ is known then the MLE of σ^2 is

$$\widehat{\sigma_0^2} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$$

- We can show that

$$\frac{n\,\widehat{\sigma_0^2}}{\sigma^2} \sim \chi^2_n$$

- What is the distribution of $\widehat{\sigma_0^2}$?

Sample mean and sample variance

- Let $X_1, \ldots, X_n$ be a random sample
- The *sample mean* and the *sample variance* are defined as

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

Theorem 8.3.1

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Then $\overline{X}_n$ and S_n are independent random variables and

$$\overline{X}_n \sim N(\mu, \sigma^2/n) \quad \text{and}$$

$$\frac{n}{\sigma^2} S_n = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \sim \chi^2_{n-1}$$

Sample mean and sample variance

About Theorem 8.3.1:

- $\overline{X}_n$ and S_n are the MLE's of μ and σ^2
- $\overline{X}_n \sim N(\mu, \sigma^2/n)$ was already known
- We knew that $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 \sim \chi^2_n$. The effect of replacing μ with $\overline{X}_n$ is that the degrees of freedom go from n to $n - 1$
- Even though $\overline{X}_n$ and S_n are functions of the same random variables they are independent (only when the random sample is drawn from a normal distribution)

Example

Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ and let

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

a) Assuming $n = 16$ determine

$$P\left(\frac{1}{2}\sigma^2 \leq \hat{\sigma}^2 \leq 2\sigma^2\right)$$

b) Determine the smallest value of n so that

$$P\left(\frac{1}{2}\sigma^2 \leq \hat{\sigma}^2 \leq 2\sigma^2\right) \geq 0.9$$

The student's *t* distributions

The *t* distributions

Let $Y \sim \chi^2_m$ and $Z \sim N(0,1)$ be independent. Then the distribution of

$$X = \frac{Z}{\left(\frac{Y}{m}\right)^{1/2}}$$

is called the *t distribution with m degrees of freedom, or* t_m

- You can see where this is going: We want the distribution of $\frac{X-\mu}{\sigma}$ where the σ is replaced by the sample standard deviation.
- Introduced by W. S. Gosset, who wrote under the alias "Student"
- The legend:
  - Gosset derived the t_m distribution while working for the Guinness Brewery in Dublin. In fear of competition he was forbidden to publish his analysis of brewery data and hence he wrote under the pseudonym Student.

The student's *t* distributions

- The pdf of the t_m distribution is

$$\frac{\Gamma\left(\frac{m+1}{2}\right)}{(m\pi)^{1/2}\Gamma\left(\frac{m}{2}\right)}\left(1+\frac{x^2}{m}\right)^{-(m+1)/2} \qquad -\infty < x < \infty$$

- Tabulated in the back of the textbook

If $X \sim t_m$ then

- $E(X) = 0$ if $m > 1$, does not exist otherwise
- $\mathrm{Var}(X) = \frac{m}{m-2}$ if $m > 2$, does not exist otherwise

Connection to the normal random variables

Theorem 8.4.2

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ and let $\overline{X}_n$ be the sample mean and let

$$\sigma' = \left[\frac{\sum_{i=1}^{n} (X_i - \overline{X}_n)^2}{n-1} \right]^{1/2}$$

Then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma'} \sim t_{n-1}$$

- Note that σ' is not the MLE of σ, but

$$\sigma' = \left(\frac{n}{n-1} \right)^{1/2} \hat{\sigma}$$

- For large n, σ' and $\hat{\sigma}$ are close.

Connection to the normal and Cauchy

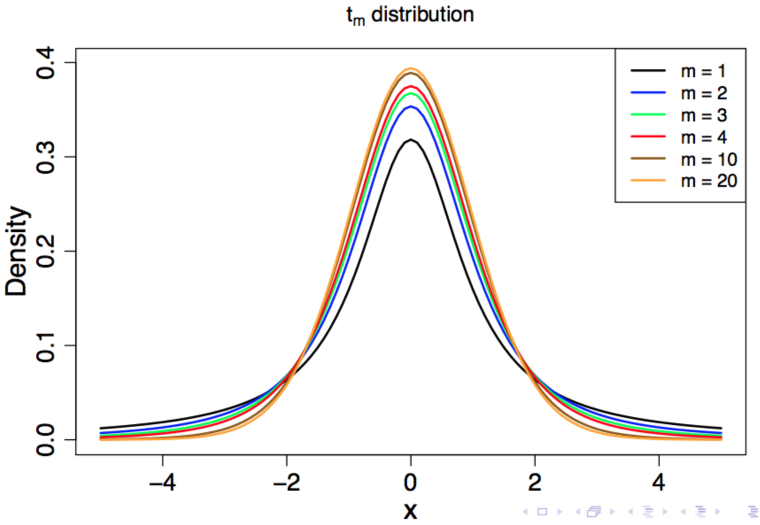
- As $m \to \infty$ the t_m approaches $N(0,1)$
- $t_1 = \mathrm{Cauchy}$:

$$f(x) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{(m\pi)^{1/2}\Gamma\left(\frac{m}{2}\right)}\left(1 + \frac{x^2}{m}\right)^{-(m+1)/2}$$

if $m = 1$:

$$f(x) = \frac{1}{\pi(1+x^2)}$$

The t_m distributions



The t_m distributions