Parametric Statistics

Convergence/Bayesian Estimation/MLE estimation

1. A random sample of n items is to be taken from a distribution with mean μ and standard deviation σ.

   (a) Use the Chebyshev inequality to determine the smallest number of items n that must be taken in order to satisfy the following relation:

   $$P(|\overline{X}_n - \mu|) \leq \frac{\sigma}{4}) \geq 0.99$$

   (b) Use the central limit theorem to determine the smallest number of items n that must be taken in order to satisfy the relation in part (a) approximately.

2. Suppose that the proportion of defective items in a large manufactured lot is 0.1. Use the central limit theorem to determine the smallest sample of items that must be taken from the lot in order for the probability to be at least 0.99 that the proportion of defective items in the sample will be less than 0.13?

3. A statistician wants to estimate the mean height h (in meters) of a population, based on n i.i.d. samples $X_1, \ldots, X_n$. He uses the sample mean $\overline{X}_n = (X_l + ... + X_n)/n$ as the estimate of h, and a rough guess of 1.0 meters for the standard deviation of the samples.

   (a) How large should n be so that the standard deviation of $\overline{X}_n$ is at most 1 centimeter?

   (b) How large should n be so that Chebyshev's inequality guarantees that the estimate is within 5 centimeters from h, with probability at least 0.99?

4. Suppose that the proportion θ of defective items in a large manufactured lot is known to be either 0.1 or 0.2, and the prior p.f. of θ is as follows:

   $$p(\theta = 0.1) = 0.7, \quad p(\theta = 0.2) = 0.3$$

   Suppose also that when eight items are selected at random from the lot, it is found that exactly two of them are defective.

   (a) Draw the prior of θ.

   (b) Determine and draw the posterior p.f. of θ.

   (c) Find the MAP and MLE estimates for θ.

5. Suppose that a regular light bulb, a long-life light bulb, and an extra-long-life light bulb are being tested. The lifetime X_1 of the regular bulb has the exponential distribution with mean θ, the lifetime X_2 of the long-life bulb has the exponential distribution with mean 2θ, and the lifetime X_3 of the extra-long-life bulb has the exponential distribution with mean 3θ. *Reminder: The exponential distribution has density* $f(x) = \lambda e^{-\lambda x}, x > 0$ *and* $E(X) = \frac{1}{\lambda}$. Determine the M.L.E. of based on the observations X_1, X_2, X_3.

6. It is not known what proportion p of the purchases of a certain brand of breakfast cereal are made by women and what proportion are made by men. In a random sample of 70 purchases of this cereal, it was found that 58 were made by women and 12 were made by men.

   (a) Find the M.L.E. of p.

   (b) Suppose that it is known that $\frac{1}{2} \leq p \leq \frac{2}{3}$. Find the M.L.E. of p.

7. Romeo and Juliet start dating, but Romeo will be late on any date by a random amount X, uniformly distributed over the interval $[0, \theta]$. The parameter θ is unknown and is modeled as the value of a random variable Θ, uniformly distributed between zero and one hour.

(a) Assuming that Romeo was late by an amount x on their first date, how should Juliet use this information to update the distribution of Θ?

(b) How should Juliet update the distribution of Θ if she observes that Romeo is late by $x_1, \dots, x_n$ on the first n dates? Assume that given θ, $X_1, \dots, X_n$ are uniformly distributed between zero and θ and are conditionally independent.

(c) Find the MAP estimate of θ based on the observation $X = x$.

(d) Find the MLE estimate of θ based on the observation $X = x$.

Table 1: Part of the table of the Standard Normal distribution function $\Phi(x) = \int_{-\infty}^{x} \frac{1}{(2\pi)^{1/2}} exp(-\frac{1}{2}u^2)du$.
Example: $\Phi(2.61) = 0.99547$

z	0	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
2,2	0,9861	0,98645	0,98679	0,98713	0,98745	0,98778	0,98809	0,9884	0,9887	0,98899
2,3	0,98928	0,98956	0,98983	0,9901	0,99036	0,99061	0,99086	0,99111	0,99134	0,99158
2,4	0,9918	0,99202	0,99224	0,99245	0,99266	0,99286	0,99305	0,99324	0,99343	0,99361
2,5	0,99379	0,99396	0,99413	0,9943	0,99446	0,99461	0,99477	0,99492	0,99506	0,9952
2,6	0,99534	0,99547	0,9956	0,99573	0,99585	0,99598	0,99609	0,99621	0,99632	0,99643
2,7	0,99653	0,99664	0,99674	0,99683	0,99693	0,99702	0,99711	0,9972	0,99728	0,99736
2,8	0,99744	0,99752	0,9976	0,99767	0,99774	0,99781	0,99788	0,99795	0,99801	0,99807
2,9	0,99813	0,99819	0,99825	0,99831	0,99836	0,99841	0,99846	0,99851	0,99856	0,99861
3	0,99865	0,99869	0,99874	0,99878	0,99882	0,99886	0,99889	0,99893	0,99896	0,999