Σύνολο Ασκήσεων 3

$$\begin{rcases} fg=0 \\ h=0 \end{rcases} \Leftrightarrow \begin{Bmatrix} f=0 \\ h=0 \end{Bmatrix} \cup \begin{Bmatrix} g=0 \\ h=0 \end{Bmatrix}$$

1γ) $V = \mathbb{V}(\langle x^2+y^2-1,\ x^2-z^2-1 \rangle) \subseteq \mathbb{C}^3$

$$\begin{rcases} x^2+y^2-1=0 \\ x^2-z^2-1=0 \end{rcases} \rightsquigarrow \begin{matrix} y^2+z^2=0 \\ x^2+y^2-1=0 \end{matrix} \rightsquigarrow$$

$$V = \mathbb{V}(\langle \underset{\underset{(z+iy)(z-iy)}{\parallel}}{y^2+z^2},\ x^2+y^2-1 \rangle) = \underbrace{\mathbb{V}(\langle z+iy,\ x^2+y^2-1 \rangle)}_{(1)} \cup \underbrace{\mathbb{V}(\langle z-iy,\ x^2+y^2-1 \rangle)}_{(2)}$$

$(1) \leftarrow$ ανάγωγα $\rightarrow (2)$

(V ανάγωγο $\Leftrightarrow$ $\mathbb{I}(V)$ πρώτο ιδεώδες)

$I = \langle z+iy,\ x^2+y^2-1 \rangle$ πρώτο ιδεώδες :

$$\mathbb{C}[x,y,z] / I \cong \frac{\mathbb{C}[x,y,z]/\langle z+iy \rangle}{I/\langle z+iy \rangle}$$

$$\mathbb{C}[x,y,z]/\langle z+iy \rangle \cong \mathbb{C}[x,y] \qquad \frac{I}{\langle z+iy \rangle} \cong \langle x^2+y^2-1 \rangle$$

$$\cong \mathbb{C}[x,y] / \langle x^2+y^2-1 \rangle$$

Αρκεί ν.δ.ο. το x^2+y^2-1 ανάγωγο του $\mathbb{C}[x,y]$

Με τον ορισμό: $\mathbb{C}[x,y] = \mathbb{C}[y][x]$

$x^2+y^2-1 = (y^2-1) + x^2 \begin{cases} \overset{(0,2)}{=} \alpha(y)(b_0(y)+b_1(y)x+\underset{\neq 0}{b_2(y)}x^2) & (*) \\ \underset{(1,1)}{=} (a_0(y)+\underset{\neq 0}{a_1(y)}x)(b_0(y)+\underset{\neq 0}{b_1(y)}x). & (**) \end{cases}$

$(*)$ συντ. x^2: $1 = \underset{\substack{\| \\ c \in K \\ \neq 0}}{\alpha(y)}\ \underset{\substack{\| \\ c^{-1}}}{b_2(y)}$. $\rightsquigarrow$ $\alpha(y) = c$ τετριμμένη ανάλυση

$(**)$ ομοίως . . . .

⑥ $R =$ α.π. $K = q.f(R)$: a) : $a \in K$ τότε $(R:a) = \{t \in R : ta \in R\}$ ιδεώδες

b) $a \in R_P \Leftrightarrow (R:a) \not\subseteq P$

$\Rightarrow$: $a = \dfrac{r}{s \notin P} \Rightarrow s \cdot a = r \Rightarrow s \in (R:a),\ s \notin P$. ✓

$\Leftarrow$ Παρε $t \in (R:a)$, $t \notin P \Rightarrow t \cdot a = r \in R \Rightarrow a = \dfrac{r}{\underset{\substack{\| \\ s}}{t} \notin P} \in R_P$

Συμβ. R αι. πρ., $S \subset R$ υποσύνολο : $S^{-1}R$: $\dfrac{a}{s} = \dfrac{a'}{s'} \Leftrightarrow \exists s'' (as'-a's) = 0 \overset{S \not\ni 0}{\Longleftrightarrow} as'-a's = 0$

γ) $R = \bigcap_{p \in \max\operatorname{Spec} R!} R_p \subseteq K \left(R_p = \left\{ \frac{r \in R}{s \notin p} \right\} \subseteq q.f.(R) = K \right)$

$\subseteq$ $r = \frac{r}{1} \overset{?}{\in} R_p, \forall p$ ok διότι $1 \notin p \Rightarrow r \in \bigcap_{p \in \max\operatorname{Spec} R} R_p$

$1 \cdot r \in R$: $(R:r) = R \overset{(β)}{\rightsquigarrow} R_\cap$

$\nexists p, \forall p$ max

$1 \in R$, $1 \notin p$ $\forall p$ max $\rightsquigarrow r \in R_p$..,

$\supseteq$: $a \in R_p, \forall p^{max} \Rightarrow a \in R$:

Έστω $(R:a) \subseteq m \subseteq R$ αντίφαση μ-

Όμως $a \in R_m \overset{(β)}{\Longrightarrow} (R:a) \not\subseteq m$.

Άρα $(R:a) = R \Rightarrow 1 \cdot a \in R \Rightarrow a \in R$.

б) $a \neq 0$, $S = \{a^n, n \in \mathbb{N}_0\}$, $\supseteq L = \{a^{2n}, n \in \mathbb{N}_0\}$

$S^{-1}R \cong L^{-1}R$, :

$S^{-1}R = \{\frac{r}{a^n}, r \in R, n \in \mathbb{N}_0\} \subseteq K$, $L^{-1}R = \{\frac{r}{a^{2n}}, r \in R, n \in \mathbb{N}_0\} \subseteq K$

$L^{-1}R \subseteq S^{-1}R$

$$\begin{tikzcd}
 & R \arrow[r, hook, "\varphi_1"] \arrow[d, equal] \arrow[dl, "\psi"'] & S^{-1}R \arrow[d, dashed] \arrow[ll, dashed, bend right, "\exists \tilde{\psi}"'] \\
T & R \arrow[l] \arrow[r, hook, "\varphi_2"] & L^{-1}R
\end{tikzcd}$$

$\varphi_1(L) \subseteq S$

$\psi(L) \subset U(T) \Rightarrow \psi(S) \subset U(T)$

$S^{-1}R = \{\frac{r}{a^n} = \frac{r \cdot a^n}{a^{2n}} \in L^{-1}R\} \subseteq K$

$R = \mathbb{Z}$, $S = \{3^n, n \in \mathbb{N}_0\}$, $L = \{3^{2n}, n \in \mathbb{N}_0\}$:

$\left[\frac{1}{3}\right] = \left[\frac{3}{3^2}\right]$.

$S^{-1}\mathbb{Z} = \{\frac{m}{3^n}, m \in \mathbb{Z}\}$

$L^{-1}\mathbb{Z} = \{\frac{m}{3^{2n}}, m \in \mathbb{Z}\}$

(4) (α) $I \subseteq p \trianglelefteq R$ (πρώτο) :

$$R \xrightarrow{j} R_p$$

$$I \qquad I^e = \left\{ \frac{a}{s},\ a \in I,\ s \notin p \right\}$$

μπορούν να φύγουν τέτοια

$$R_p / I^e \cong \left(R/I \right)_{\bar{p}} \quad , \quad R \xrightarrow{\pi} R/I, \quad p \longmapsto \pi(p) = \bar{p}.$$

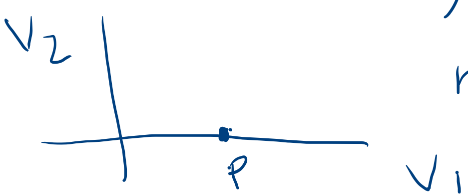
$$\bar{a} \in \bar{p} \Longleftrightarrow a \in p.$$

$\exists \rightarrow$ (επί, πυρήνας I^e.

$$\begin{tikzcd}
R_p \arrow[r, dashed, "\exists"] & \left(R/I\right)_{\bar{p}} \\
R \arrow[u] \arrow[r] \arrow[ur] & R/I \arrow[u]
\end{tikzcd}$$

$$a \qquad\qquad \bar{a} \mapsto \frac{\bar{a}}{\bar{1}}$$

β) $V = V_1 \cup V_2$, $p \in V_1$ $\qquad k[V] = k[x_1, \ldots, x_n] / \mathbb{I}(V)$

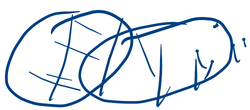


$$m = \mathbb{I}(\{p\}) \qquad k[V]_m \cong \underbrace{k[V_1]}_{= k[x_1, \ldots, x_n] / \mathbb{I}(V_1)}{}_{\bar{m}}.$$

$$R_p/I^e \cong (R/I)_{\bar{p}}$$

$$k[V]_{\bar{m}} = \left(k[x_1,\dots,x_n]/I(V)\right)_{\bar{m}} \cong k[x_1,\dots,x_n]_m \big/ I(V)^e$$

$V = V_1 \cup V_2$: $I(V) = I(V_1) \cap I(V_2)$



$$I(V)^e = I(V_1)^e \cap I(V_2)^e \quad (8\ \gamma 1)$$

$$\cong k[x_1,\dots,x_n]_m \big/ I(V_1)^e \cap I(V_2)^e = k[x_1,\dots,x_n]_m \big/ I(V_1)^e \cong k[V_1]_{\bar{m}_1}$$

$$k[x_1,\dots,x_n] \longrightarrow k[x_1,\dots,x_n]_{m = I(p)}$$

$$I(V_2) \longrightarrow I(V_2)^e = k[x_1,\dots,x_n]$$

shift to !:

6 δ) : $S^{-1}R$: $R\times S$ ορίζουμε $\sim$: $(r,s)\sim(r',s')$ $\Leftrightarrow$ $rs'-r's=0$

$S=\{a^n, n\in\mathbb{N}\}$

$$S^{-1}R = \left\{ \frac{r}{s} = [(r,s)] \right\}$$

$T \hookrightarrow R$

$T' \cup$

$$\frac{r}{s} = \frac{rs'}{ss'}$$

$$\begin{array}{ccc} S^{-1}R & \xhookrightarrow{i} & K \\ r/s & \longmapsto & r/s \\ L^{-1}R & \xhookrightarrow{j} & K \\ r/l & \longmapsto & r/l \end{array}$$

$\operatorname{Im} i = \operatorname{Im} j$.

$L=\{a^{2n}, n\in\mathbb{N}\}$

$L^{-1}R$: $R\times L$ ορίζουμε $(r,l)\sim(r',l')$ $\Leftrightarrow$ $rl'-r'l=0$

$$L^{-1}R = \left\{ \frac{r}{l} = [r,l] \right\}.$$

$$\begin{array}{ccc} L^{-1}R & \xrightarrow{\varphi} & S^{-1}R \\ \frac{r}{a^{2n}} & \longmapsto & \frac{r}{a^{2n}} \end{array}$$

ομομ. δακτυλίων

$\ker\varphi = (0)$

Επί?

$$\frac{r}{a^n} = \frac{ra^n}{a^{2n}} = \varphi\left(\frac{ra^n}{a^{2n}}\right)$$