① $R=\mathbb{Z}[i\sqrt{5}]$

$\langle 2\rangle$ πρώτο, $\langle 2, 1+i\sqrt{5}\rangle = J$

$\longrightarrow \quad \underbrace{(1+i\sqrt{5})}_{\not\in \langle 2\rangle}{}^2 = -4+2i\sqrt{5} \in \langle 2\rangle$

$\underset{\not\in\langle 2\rangle}{(1+i\sqrt{5})}\underset{\not\in\langle 2\rangle}{(1-i\sqrt{5})} = \underset{2\cdot 3}{\overset{}{6}} \in \langle 2\rangle$

$$\frac{R/I}{J/I} \cong R/J$$

$\longrightarrow \quad \langle 1+i\sqrt{5}\rangle = I \subseteq J \subseteq R$

$R/I \underset{A}{\cong} \mathbb{Z}_6$: $\mathbb{Z} \longrightarrow R/J$

$\mathbb{Z}_6/\langle 2\rangle \cong \mathbb{Z}_2$ σώμα.

$R \longrightarrow \mathbb{Z}_2$

$n+im\sqrt{5} \to (n+m) \bmod 2$

$$\mathbb{Z}[i\sqrt{5}] \cong \mathbb{Z}[x]/\langle x^2+5\rangle \longleftrightarrow \mathbb{Z}[x]$$

$$\langle 2\rangle \longleftrightarrow \ ?\ . \qquad\qquad ?$$

$$\langle 2, 1+i\sqrt{5}\rangle \longleftrightarrow \ ? \qquad \longleftrightarrow \ ?$$

(2)

$I+J=R \qquad IJ = I\cap J.$

$k+\lambda = 1 \qquad \overset{\subseteq}{\supseteq}$

$b,\ \underset{\parallel}{a} \in I\cap J$

$a\cdot 1 = \dots$

(3) $\langle x^5 \rangle = \langle f_1(x) \rangle \cap \langle f_2(x) \rangle \Rightarrow f_1, f_2 \mid x^5$

$\langle x^5 \rangle \subset R[x]$

$\langle f_1(x) \rangle \supsetneq \langle x^5 \rangle$, $\quad \langle f_2(x) \rangle \supsetneq \langle x^5 \rangle$

$f_1 = x^i, \quad f_2 = x^j, \qquad i, j \le 5$

$\underbrace{\langle x^i \rangle \cap \langle x^j \rangle}_{=\langle x^i \rangle}$ $\qquad i \le j$

… … άτοπο..

(4) $R = \mathbb{C}[x, y, z]$, $\quad I = \langle x^4, x^2y^5, y^3z^4 \rangle$: $\mathrm{Rad}\, I = ?$

Μάντεψε υποψήφιος: $J = \langle x, yz \rangle$

$(yz)^4 = y \cdot \underbrace{(y^3 z^4)}_{\in I}$

- $J \subseteq \mathrm{Rad}\, I$
- $I \subseteq J \subseteq \mathrm{Rad}\, I \Rightarrow \mathrm{Rad}\, I \subseteq \underset{= J}{\mathrm{Rad}\, J} \subseteq \mathrm{Rad}\, I$

αν J είναι τον

$J = \langle x, yz\rangle \subseteq \mathbb{C}[x,y,z]$. είναι ριζικό?

($I \trianglelefteq R$: $I \overset{\subseteq}{\underset{\supseteq}{=}} \mathrm{Rad}\, I$

αν $a^n \in I \Rightarrow a \in I$)

$\langle 0\rangle \subseteq R/I$: $\bar{a}^n = 0 \Rightarrow a = 0$

$$\mathbb{C}[x,y,z] \big/ \langle x, yz\rangle \cong \mathbb{C}[y,z] \big/ \langle yz\rangle$$

$$\| $$

$$\dfrac{\mathbb{C}[x,y,z]/\langle x\rangle}{\langle x, yz\rangle/\langle x\rangle}$$

$\Uparrow$

Δείξτε ότι δεν έχει μηδενοδύναμα

$\overline{f(y,z)}^{\,n} = \bar{0} \Leftrightarrow f(x,y)^n \in \langle yz\rangle$

$yz \mid f(x,y)^n \Rightarrow \dots$

(5) $f(x) = x^3 + ax^2 + bx + c \in \mathbb{Z}[x]$, p = πρώτος

$P = \langle f(x), p\rangle$ πρώτο $\Leftrightarrow \overline{f(x)} \in \mathbb{Z}_p[x]$ δεν έχει ρίζα.

$\Updownarrow$ $\qquad\qquad \mathbb{Z}_p[x] = \mathbb{Z}[x]/?$

$\mathbb{Z}[x]/P$ ακέραια

$$\|$$

$$\dfrac{\mathbb{Z}[x]/}{P}$$

← δοκιμάστε τις 2 επιλογές!

⑥ $\varphi : R \to S$

$S \trianglerighteq Q$ πρώτο

$\varphi^{-1}(Q) = Q^c$

$\hookrightarrow$ πρώτο.

⑦ $I = \langle x-2, y-3 \rangle \trianglelefteq \mathbb{R}[x,y]$ μέγιστο ?

$\ker\varphi = \langle x-2 \rangle$

$\varphi : \mathbb{R}[x,y] \xrightarrow{ev_2} \mathbb{R}[y]$

$f(x,y) \to f(2,y)$

$$\mathbb{R}[x,y]/I \cong \frac{\mathbb{R}[x,y]/\langle x-2\rangle}{I/\langle x-2\rangle} \cong \frac{\mathbb{R}[y]}{\langle y-3\rangle} \cong \mathbb{R}.$$

$\mathbb{R}[x,y] = \underbrace{\mathbb{R}[y]}[x]$ $\quad f(x) \,\big|\, x-2$

$f(x,y) = (x-2)\cdot q(x,y) + r(y)$

$f(x,y) \in \ker\varphi \iff f(2,y) = 0, \ \forall y$

το υπόλοιπο! $\iff r(y) = 0, \ \forall y$

$\psi : \mathbb{R}[x,y] \longrightarrow \mathbb{R}$

$f(x,y) \longrightarrow f(2,3)$

(8) $I \trianglelefteq R$: $\exists\, p \in \operatorname{Spec} R : I \subseteq p$ ελαχιστικό.

$A = \{p \in \operatorname{Spec} R : I \subseteq p\}$, "$\supseteq$"

- $A \neq \emptyset$: θεώρημα

- $p_1 \supseteq p_2 \supseteq \dots$ $\quad$ $\forall i$ $p_i \supseteq J = \bigcap_i p_i$

$I \subseteq J$: με εις άτοπο : ...

(9) $\forall r \in R$, $r^n = r$, κάποιο $n > 1$, τότε κάθε πρώτο ιδεώδες είναι μέγιστο.

$\max\operatorname{Spec} R \subseteq \operatorname{Spec} R$: $p \in \operatorname{Spec} R$:

$$r \cdot (r^{n-1} - 1) = 0 \in p \quad \begin{cases} r \in p : \\ r^{n-1} - 1 \in p \end{cases}$$

$p \subsetneq J = R$

$r \in J \setminus p \Rightarrow r^{n-1} - 1 \in p \;:\; 1 = \overset{p \subseteq J}{\overset{\Downarrow}{a}} + \overset{J}{\overset{\Downarrow}{r^{n-1}}} \in J$

$\rightsquigarrow R/p \ni \overline{r} \neq \overline{0} \Rightarrow \overline{r}$ αντιστρέψιμο.