UNIVERSITY OF CRETE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS
NUMBER THEORY - MEM204 (SPRING SEMESTER 2019-20)
LECTURER: G. KAPETANAKIS

2nd exercise set - Answers

Exercise 1. Show that $\mu^{-1} = \nu$, where $\nu(n) = 1$ for all n.

Answer. We know that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 0 \\ 0, & n > 0. \end{cases} \tag{1}$$

Now, notice that $\mu(1) = 1$, hence $\exists \mu^{-1}$. We will prove that $\mu^{-1}(n) = 1$ for all n, using (strong) induction on n.

- For $n = 1$, $\mu^{-1}(1) = \frac{1}{\mu(1)} = 1$.
- Assume that $\mu^{-1}(n) = 1$ for all $n < k$, where $k \geq 2$ (I.H.).
- Let $k = p_1^{k_1} \cdots p_r^{k_r}$ be the prime factorization of k. We have that

$$\begin{aligned} \mu^{-1}(k) &= -\sum_{\substack{d|k \\ d\neq k}} \mu\left(\frac{k}{d}\right) \mu^{-1}(d) \overset{\text{I.H.}}{=} -\sum_{\substack{d|k \\ d\neq k}} \mu\left(\frac{k}{d}\right) \\ &= -\sum_{\substack{d|k \\ d\neq 1}} \mu(d) = 1 - \sum_{d|k} \mu(d) \overset{(1)}{=} 1. \end{aligned}$$

□

Exercise 2. Show that for every $n \geq 1$, $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$.

Answer. The result follows immediately as a combination of the following facts:

1. Let $0 \leq r \leq 3$ be the remainder of the euclidean division of $n+3$ by 4. Then $4 \mid n+3-r$ and $n+3-r = n, n+1, n+2$ or $n+3$.
2. If $4 \mid k$, then k is not square-free, i.e., $\mu(k) = 0$. □

Exercise 3. Let p be a prime. Show that

$$\sum_{d|n} \mu(d)\mu(\gcd(p,d)) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n = p^a,\ a \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Answer. We consider the following cases:

1. If $n = 1$, then, clearly, $\sum_{d|n} \mu(d)\mu(\gcd(p,d)) = 1$.

2. If $n > 1$ and $p \nmid n$, then $\forall\, d \mid n$, we have that $\gcd(p, d) = 1 \Rightarrow \mu(\gcd(p, d)) = 1$. It follows that

$$\sum_{d|n} \mu(d)\mu(\gcd(p, d)) = \sum_{d|n} \mu(d) \overset{(1)}{=} 0.$$

3. If $n > 1$, $p \mid n$ and $n \neq p^a$. Then we write $n = p^b m$, where $m > 1$, $b \geq 1$ and $(m, p) = 1$. It follows that

$$\begin{aligned}\sum_{d|n} \mu(d)\mu(\gcd(p, d)) &= \sum_{\substack{d|n \\ p\nmid d}} \mu(d)\mu(\gcd(p, d)) + \sum_{\substack{d|n \\ p|d}} \mu(d)\mu(\gcd(p, d)) \\ &= \sum_{d|m} \mu(d) + \mu(p)^2 \sum_{d|m} \mu(d) \overset{(1)}{=} 0.\end{aligned}$$

4. If $n = p^a$, $a \geq 1$, then

$$\begin{aligned}\sum_{d|n} \mu(d)\mu(\gcd(p, d)) &= \sum_{i=0}^{a} \mu(p^i)\mu(\gcd(p, p^i)) \\ &= \mu(1)^2 + \mu(p)^2 + \sum_{i=2}^{a} \mu(p^i)\mu(p) \\ &\overset{(1)}{=} 1^2 + (-1)^2 + 0 = 2.\end{aligned}$$

□

Exercise 4. Show that for every $n > 2$, $\phi(n)$ is even.

Answer. We take two cases:

1. If $n = 2^a$, where $a \geq 2$. Then $\phi(n) = 2^{a-1}$, where $a - 1 \geq 1$, so $\phi(n)$ is even.

2. If n is divided by an odd prime p, then we easily see that $p - 1 \mid \phi(n)$. Thus, since $p - 1$ is even, so is $\phi(n)$. □

Exercise 5. How many numbers $1 \leq k \leq 3600$ have a non-trivial common factor with 3600?

Answer. First, notice that $3600 = 2^4 3^2 5^2$. Also, in total, we have 3600 numbers in the interval $1 \leq k \leq 3600$. Among them, there are

$$\phi(3600) = 2^{4-1}3^{2-1}5^{2-1}(2-1)(3-1)(5-1) = 2^3 \cdot 3 \cdot 5 \cdot 1 \cdot 2 \cdot 4 = 960$$

numbers that are co-prime to 3600. It follows that the remaining $3600 - 960 = 2640$ numbers in the interval have a non-trivial common factor with 3600. □

Exercise 6. Show that $m \mid n \Rightarrow \phi(m) \mid \phi(n)$.

Answer. Since $m \mid n$, we can assume that if

$$m = p_1^{m_1} \cdots p_k^{m_k},$$

where $m_i \geq 1$, is the prime factorization of m, then the prime factorization of n is of the form

$$n = p_1^{n_1} \cdots p_k^{n_k} p_{k+1}^{n_{k+1}} \cdots p_\ell^{n_\ell},$$

where $n_i \geq m_i$, for $1 \leq i \leq k$ and $n_i \geq 1$, for $k+1 \leq i \leq \ell$.

It follows that

$$\phi(m) = p_1^{m_1-1} \cdots p_k^{m_k-1}(p_1 - 1) \cdots (p_k - 1)$$

and

$$\phi(n) = p_1^{n_1-1} \cdots p_k^{n_k-1}(p_1 - 1) \cdots (p_k - 1) p_{k+1}^{n_{k+1}-1} \cdots p_\ell^{n_\ell-1}(p_{k+1} - 1) \cdots (p_\ell - 1).$$

The result follows immediately from the fact that $n_i \geq m_i$, for $1 \leq i \leq k$. □

Exercise 7. Show that if m and n have the same prime factors (possibly in different powers), then $n\phi(m) = m\phi(n)$.

Answer. Let $m = p_1^{m_1} \cdots p_k^{m_k}$ and $n = p_1^{n_1} \cdots p_k^{n_k}$, where $n_i, m_i \geq 1$ be the prime factorizations of m and n. Then

$$\begin{aligned} n\phi(m) &= p_1^{n_1} \cdots p_k^{n_k} p_1^{m_1-1} \cdots p_k^{m_k-1}(p_1 - 1) \cdots (p_k - 1) \\ &= p_1^{m_1} \cdots p_k^{m_k} p_1^{n_1-1} \cdots p_k^{n_k-1}(p_1 - 1) \cdots (p_k - 1) \\ &= m\phi(n). \end{aligned}$$

□

Exercise 8. Find all n such that $\phi(n) = \frac{n}{2}$.

Answer. Let $n = p_1^{n_1} \cdots p_k^{n_k}$, where $n_i \geq 1$ be the prime factorization of n. Then $\phi(n) = n/2$ implies

$$p_1^{n_1-1} \cdots p_k^{n_k-1}(p_1 - 1) \cdots (p_k - 1) = \frac{p_1^{n_1} \cdots p_k^{n_k}}{2},$$

that is,

$$2(p_1 - 1) \cdots (p_k - 1) = p_1 \cdots p_k.$$

The RHS of the above equation is square-free, so the same should hold for the RHS. However, this is only possible if $(p_1 - 1) \cdots (p_k - 1) = 1$, i.e., if $n = 2^a$, $a \geq 1$. Moreover, we easily verify that $\phi(2^a) = 2^{a-1} = \frac{2^a}{2}$. To sum up, $\phi(n) = \frac{n}{2}$ if and only if $n = 2^a$ for some $a \geq 1$. □

Exercise 9. Find all n such that $\phi(n) = 12$.

Answer. Write $n = p_1^{n_1} \cdots p_k^{n_k}$, where $p_i \neq p_j$ and $n_i \geq 1$. Then

$$\phi(n) = \big(p_1^{n_1-1}(p_1 - 1)\big) \cdots \big(p_k^{n_k-1}(p_k - 1)\big).$$

It follows that $\phi(n) = 12$ implies that for every i, we have that $\phi(p_i^{n_i}) = p_i^{n_i-1}(p_i - 1) \mid 12$. This suggests that it suffices to cycle through the divisors d of 12 and try to find all the possible pairs (p_i, n_i), such that $\phi(p_i^{n_i}) = d$ and then see which of those pairs can be combined with each other, in order to get the required product 12. Note that two pairs with the same prime cannot co-exist.

The divisors of 12 are: $\{1, 2, 3, 4, 6, 12\}$. We examine each divisor separately:

$d = 1$. Here, we have the pair $(p_i, n_i) = (2, 1)$ (that is, $\phi(2) = 1$).

$d = 2$. Here, we have the pairs $(p_i, n_i) = (2, 2)$ and $(3, 1)$ (that is, $\phi(4) = \phi(3) = 2$).

$d = 3$. This is impossible by Exercise 4.

$d = 4$. Here, we have the pairs $(p_i, n_i) = (2, 3)$ and $(5, 1)$.

$d = 6$. Here, we have the pairs $(p_i, n_i) = (3, 2)$ and $(7, 1)$.

$d = 12$. Here, we have the pair $(13, 1)$.

We have the following possibilities, corresponding to the largest divisor appearing.

1. If the largest divisor is $d = 12$, then $(13, 1)$ appears and it may appear on its own, or along with the pair $(2, 1)$. We get two corresponding numbers, $n = 13$ and $n = 26$.

2. If the largest divisor is $d = 6$, then $d = 2$ should appear and $d = 1$ *may* appear. It follows that we may have 8 options (2 choices for $d = 6$, 2 choices for $d = 2$ and 2 choices for adding $d = 1$ or not. In total $2 \cdot 2 \cdot 2 = 8$ options). In particular, we have

   (a) $\{(3, 2), (2, 2), (2, 1)\}$
   (b) $\{(3, 2), (2, 2)\}$
   (c) $\{(3, 2), (3, 1), (2, 1)\}$
   (d) $\{(3, 2), (3, 1)\}$
   (e) $\{(7, 1), (2, 2), (2, 1)\}$
   (f) $\{(7, 1), (2, 2)\}$
   (g) $\{(7, 1), (3, 1), (2, 1)\}$
   (h) $\{(7, 1), (3, 1)\}$

   Since two coexisting pairs cannot share the same first coordinate, we exclude the options $\{(3, 2), (2, 2), (2, 1)\}$, $\{(3, 2), (3, 1), (2, 1)\}$, $\{(3, 2), (3, 1)\}$ and $\{(7, 1), (2, 2), (2, 1)\}$, so we are left with 4 options. They correspond to the numbers: 36, 28, 42 and 21.

3. If the largest divisor is $d = 4$, then $d = 3$ should appear, but this is impossible.

4. The largest divisor cannot be $d = 3$, as $d = 3$ is impossible.

5. Finally, clearly, the largest divisor cannot be 2 or 1.

All in all, there are six choices for n, that is, 13, 26, 36, 28, 42 and 21. □

Exercise 10. Find all n such that $\sigma(n) = 12$.

Answer. It is clear that $\sigma(n) \geq n + 1 \iff n \leq \sigma(n) - 1$. It follows that it suffices to check the numbers $n \leq 11$. A quick computation reveals:

1. $\sigma(1) = 1$.
2. $\sigma(2) = 3$.
3. $\sigma(3) = 4$.
4. $\sigma(4) = 7$.
5. $\sigma(5) = 6$.
6. $\sigma(6) = 12$.
7. $\sigma(7) = 8$.
8. $\sigma(8) = 15$.
9. $\sigma(9) = 13$.
10. $\sigma(10) = 17$.
11. $\sigma(11) = 12$.

So, $n = 6$ or $n = 11$. □

Exercise 11. Find all n such that $\tau(n) = 12$.

Answer. Write $n = p_1^{n_1} \cdots p_k^{n_k}$, where $p_i \neq p_j$ and $n_i \geq 1$. Then, we have that

$$\tau(n) = (n_1 + 1) \cdots (n_k + 1).$$

W.l.o.g. assume that the numbers $(n_1 + 1), \ldots, (n_k + 1)$ are in descending order. Then each of them is a divisor > 1 of 12. Then we have the following options:

1. $n_1 = 11$.
2. $n_1 = 5, n_2 = 1$.
3. $n_1 = 3, n_2 = 2$.
4. $n_1 = 2, n_2 = 1, n_3 = 1$.

It follows that $\tau(n) = 12$ iff n is factorized into primes in one of the following ways

1. $n = p_1^{11}$,
2. $n = p_1^5 p_2$,
3. $n = p_1^3 p_2^2$ or
4. $n = p_1^2 p_2 p_3$,

where the numbers p_1, p_2 and p_3 are distinct primes. □

Exercise 12. Define $\sigma_k(n) = \sum_{d|n} d^k$. Show that σ_k is multiplicative.

Answer. It is obvious that $f(n) = n^k$ is multiplicative. The desired result follows. □

Exercise 13. Prove the following identities:

1. $\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$.
2. $\sum_{d|n} \sigma(d) = n \sum_{d|n} \frac{\tau(d)}{d}$.

3. $n \cdot \sum_{d|n} \frac{\sigma(d)}{d} = \sum_{d|n} d \cdot \tau(d)$.

Answer. We begin with the **first** item. Note that $f(n) = \frac{1}{n}$ is multiplicative, hence $\sum_{d|n} \frac{1}{d}$ is multiplicative. Moreover, $\sigma(n)$ is multiplicative, so, $\sigma(n)f(n) = \frac{\sigma(n)}{n}$ is multiplicative. It follows that it suffices to show that

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n},$$

for $n = p^a$, where p is a prime and $a \geq 1$. Now, we have that

$$\sum_{d|p^a} \frac{1}{d} = \sum_{i=0}^{a} \frac{1}{p^i} = \frac{1}{p^a} \sum_{i=0}^{a} p^{a-i} = \frac{1}{p^a} \sum_{i=0}^{a} p^i = \frac{\sigma(p^a)}{p^a}.$$

Next, we focus on the **second** item. Using similar arguments as before, one can see that $\sum_{d|n} \sigma(d)$ and $n \sum_{d|n} \frac{\tau(d)}{d}$ are both multiplicative. It follows that it suffices to show the equality when $n = p^a$. We have that

$$\sum_{d|p^a} \sigma(d) = \sum_{i=0}^{a} \sigma(p^i) = \sum_{i=0}^{a} \sum_{j=0}^{i} p^j = \sum_{i=0}^{a} (a-i+1)p^i,$$

and

$$p^a \sum_{d|p^a} \frac{\tau(d)}{d} = p^a \sum_{i=0}^{a} \frac{\tau(p^i)}{p^i} = \sum_{i=0}^{a} (i+1)p^{a-i}.$$

The result follows.

Finally, we focus on the **third** item. Using similar arguments as before, one can see that the two functions are multiplicative. It follows that it suffices to show the equality when $n = p^a$. We have that

$$p^a \sum_{d|p^a} \frac{\sigma(d)}{d} = p^a \sum_{i=0}^{a} \frac{\sigma(p^i)}{p^i} = \sum_{i=0}^{a} \sum_{j=0}^{i} p^{a-i+j} = \sum_{i=0}^{a} (i+1)p^i,$$

and

$$\sum_{d|p^a} d\tau(d) = \sum_{i=0}^{a} p^i \tau(p^i) = \sum_{i=0}^{a} (i+1)p^i.$$

The result follows. □

Exercise 14. If n is a perfect number, show that $\sum_{d|n} \frac{1}{d} = 2$.

Answer. If n is perfect, then

$$\sigma(n) = 2n \Rightarrow \sum_{d|n} d = 2n \Rightarrow \sum_{d|n} \frac{n}{d} = 2n \Rightarrow n \left(\sum_{d|n} \frac{1}{d} \right) = 2n \Rightarrow \sum_{d|n} \frac{1}{d} = 2.$$ □