MEM204-Number Theory

1st virtual lecture

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2nd set - Answers

Exercise 2

Exercise

Show that for every $n \geq 1$, $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$.

Answer

The result follows immediately as a combination of the following facts:

1. *Let* $0 \leq r \leq 3$ *be the remainder of the euclidean division of* $n+3$ *by* 4. *Then* $4 \mid n+3-r$ *and* $n+3-r = n, n+1, n+2$ *or* $n+3$.
2. *If* $4 \mid k$, *then* k *is not square-free, i.e.,* $\mu(k) = 0$.

Exercise 3

Exercise

Let p be a prime. Show that

$$\sum_{d|n} \mu(d)\mu(\gcd(p,d)) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n = p^a,\ a \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 3

We consider the following cases:

- If $n = 1$, then, clearly,

$$\sum_{d|n} \mu(d)\mu(\gcd(p,d)) = 1.$$

- If $n > 1$ and $p \nmid n$, then $\forall\, d \mid n$, we have that $\gcd(p,d) = 1 \Rightarrow \mu(\gcd(p,d)) = 1$. It follows that

$$\sum_{d|n} \mu(d)\mu(\gcd(p,d)) = \sum_{d|n} \mu(d) = 0.$$

Exercise 3

- If $n > 1$, $p \mid n$ and $n \neq p^a$. Then we write $n = p^b m$, where $m > 1$, $b \geq 1$ and $(m, p) = 1$. It follows that

$$\begin{aligned}
&\sum_{d|n} \mu(d)\mu(\gcd(p,d)) \\
&= \sum_{d|n,\ p\nmid d} \mu(d)\mu(\gcd(p,d)) + \sum_{d|n,\ p|d} \mu(d)\mu(\gcd(p,d)) \\
&= \sum_{d|m} \mu(d) + \mu(p)^2 \sum_{d|m} \mu(d) = 0.
\end{aligned}$$

Exercise 3

- If $n = p^a$, $a \geq 1$, then

$$\sum_{d|n} \mu(d)\mu(\gcd(p,d)) = \sum_{i=0}^{a} \mu(p^i)\mu(\gcd(p,p^i))$$

$$= \mu(1)^2 + \mu(p)^2 + \sum_{i=2}^{a} \mu(p^i)\mu(p)$$

$$= 1^2 + (-1)^2 + 0 = 2.$$

Exercise 4

Exercise

Show that for every $n > 2$, $\varphi(n)$ is even.

Answer

We take two cases:

1. *If $n = 2^a$, where $a \geq 2$. Then $\varphi(n) = 2^{a-1}$, where $a - 1 \geq 1$, so $\varphi(n)$ is even.*
2. *If n is divided by an odd prime p, then we easily see that $p - 1 \mid \varphi(n)$. Thus, since $p - 1$ is even, so is $\varphi(n)$.*

Exercise 5

Exercise

How many numbers $1 \leq k \leq 3600$ *have a non-trivial common factor with* 3600*?*

Answer

First, notice that $3600 = 2^4 3^2 5^2$*. Also, in total, we have* 3600 *numbers in the interval* $1 \leq k \leq 3600$*. Among them, there are*

$$\varphi(3600) = 2^{4-1}3^{2-1}5^{2-1}(2-1)(3-1)(5-1) = 2^3 \cdot 3 \cdot 5 \cdot 1 \cdot 2 \cdot 4 = 960$$

numbers that are co-prime to 3600*. It follows that the remaining* $3600 - 960 = 2640$ *numbers in the interval have a non-trivial common factor with* 3600*.*

Exercise 6

Exercise

Show that $m \mid n \Rightarrow \varphi(m) \mid \varphi(n)$.

Exercise 6

Since $m \mid n$, we can assume that if

$$m = p_1^{m_1} \cdots p_k^{m_k},$$

where $m_i \geq 1$, is the prime factorization of m, then the prime factorization of n is of the form

$$n = p_1^{n_1} \cdots p_k^{n_k} p_{k+1}^{n_{k+1}} \cdots p_\ell^{n_\ell},$$

where $n_i \geq m_i$, for $1 \leq i \leq k$ and $n_i \geq 1$, for $k+1 \leq i \leq \ell$.

It follows that

$$\varphi(m) = p_1^{m_1-1} \cdots p_k^{m_k-1}(p_1 - 1) \cdots (p_k - 1)$$

and

$$\varphi(n) = p_1^{n_1-1} \cdots p_\ell^{n_\ell-1}(p_1 - 1) \cdots (p_\ell - 1).$$

The result follows immediately from the fact that $n_i \geq m_i$, for $1 \leq i \leq k$.

Exercise 7

Exercise

Show that if m and n have the same prime factors (possibly in different powers), then $n\varphi(m) = m\varphi(n)$.

Answer

Let $m = p_1^{m_1} \cdots p_k^{m_k}$ and $n = p_1^{n_1} \cdots p_k^{n_k}$, where $n_i, m_i \geq 1$ be the prime factorizations of m and n. Then

$$\begin{aligned} n\varphi(m) &= p_1^{n_1} \cdots p_k^{n_k} p_1^{m_1-1} \cdots p_k^{m_k-1}(p_1 - 1) \cdots (p_k - 1) \\ &= p_1^{m_1} \cdots p_k^{m_k} p_1^{n_1-1} \cdots p_k^{n_k-1}(p_1 - 1) \cdots (p_k - 1) \\ &= m\varphi(n). \end{aligned}$$

Exercise 8

Exercise

Find all n such that $\varphi(n) = \frac{n}{2}$.

Exercise 8

Let $n = p_1^{n_1} \cdots p_k^{n_k}$, where $n_i \geq 1$ be the prime factorization of n. Then $\varphi(n) = n/2$ implies

$$p_1^{n_1-1} \cdots p_k^{n_k-1}(p_1 - 1) \cdots (p_k - 1) = \frac{p_1^{n_1} \cdots p_k^{n_k}}{2},$$

that is,

$$2(p_1 - 1) \cdots (p_k - 1) = p_1 \cdots p_k.$$

The RHS of the above equation is square-free, so the same should hold for the RHS. However, this is only possible if $(p_1 - 1) \cdots (p_k - 1) = 1$, i.e., if $n = 2^a$, $a \geq 1$. Moreover, we easily verify that $\varphi(2^a) = 2^{a-1} = \frac{2^a}{2}$. To sum up, $\varphi(n) = \frac{n}{2}$ if and only if $n = 2^a$ for some $a \geq 1$.

Exercise 10

Exercise

Find all n such that $\sigma(n) = 12$.

Answer

It is clear that $\sigma(n) \geq n + 1 \iff n \leq \sigma(n) - 1$. *It follows that it suffices to check the numbers* $n \leq 11$. *A quick computation reveals that in the interval* $1 \leq n \leq 11$, *only* $n = 6$ *and* $n = 11$ *satisfy* $\sigma(n) = 12$.

Exercise 11

Exercise

Find all n such that $\tau(n) = 12$.

Exercise 11

Write $n = p_1^{n_1} \cdots p_k^{n_k}$, where $p_i \neq p_j$ and $n_i \geq 1$. Then, we have that

$$\tau(n) = (n_1 + 1) \cdots (n_k + 1).$$

W.l.o.g. assume that the numbers $(n_1 + 1), \ldots, (n_k + 1)$ are in descending order. Then each of them is a divisor > 1 of 12. Then we have the following options:

1. $n_1 = 11$.
2. $n_1 = 5$, $n_2 = 1$.
3. $n_1 = 3$, $n_2 = 2$.
4. $n_1 = 2$, $n_2 = 1$, $n_3 = 1$.

Exercise 11

It follows that $\tau(n) = 12$ iff n is factorized into primes in one of the following ways

1. $n = p_1^{11}$,
2. $n = p_1^5 p_2$,
3. $n = p_1^3 p_2^2$ or
4. $n = p_1^2 p_2 p_3$,

where the numbers p_1, p_2 and p_3 are distinct primes.

Exercise 14

Exercise

If n is a perfect number, show that $\sum_{d|n} \frac{1}{d} = 2$.

Answer

If n is perfect, then

$$\sigma(n) = 2n \Rightarrow \sum_{d|n} d = 2n \Rightarrow \sum_{d|n} \frac{n}{d} = 2n$$

$$\Rightarrow n\left(\sum_{d|n} \frac{1}{d}\right) = 2n \Rightarrow \sum_{d|n} \frac{1}{d} = 2.$$

Stay home, stay safe!