1. (HW 10.2) If $a \in \mathbb{R} \setminus \{0\}$ and $0 < \sigma < 1$ show that the sequence $\{an^\sigma\}$ is uniformly distributed in $[0,1]$. ($\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.)

Use Weyl's criterion. Approximate the sum $\sum_{n=1}^N e^{2\pi i k\{an^\sigma\}} = \sum_{n=1}^N e^{2\pi i k a n^\sigma}$ by the integral $\int_1^N e^{2\pi i k a x^\sigma}\, dx$ and bound their difference using the Mean Value Theorem in every interval of the form $[i, i+1]$.

Solution:

Define $f_k(x) = e^{2\pi i k a x^\sigma}$. Then $f_k'(x) = 2\pi i k a \sigma x^{\sigma-1} e^{2\pi i k a x^\sigma}$.

We must show that for all integers $k \neq 0$ we have

$$\sum_{n=1}^N e^{2\pi i k\{an^\sigma\}} = o(N), \text{ as } N \to \infty.$$

Since $\{t\} = t - \lfloor t \rfloor$ for any t it follows that the above sum is the same (we ignore the last term which is at most 1, so it cannot affect the conclusion) as

$$S_{k,N} = \sum_{n=1}^{N-1} e^{2\pi i k a n^\sigma} = \sum_{n=1}^{N-1} f_k(n)$$

We will compare this sum to the integral

$$I_{k,N} = \int_1^N e^{2\pi i k a x^\sigma}\, dx = \int_1^N f_k(x)\, dx.$$

First we compute $I_{k,N}$. It differs by a bounded quantity from $\int_0^N e^{2\pi i k a x^\sigma}\, dx$ so we compute the latter integral which is easier and we will show that it is $o(N)$ as $N \to \infty$. After the change of variable

$$y = \frac{2\pi}{N^\sigma} x^\sigma,$$

designed to lead to the interval of integration $[0, 2\pi]$, we get

$$\int_0^N e^{2\pi i k a x^\sigma}\, dx = \frac{N}{\sigma (2\pi)^{1/\sigma}} \int_0^{2\pi} e^{ikaN^\sigma y} y^{\frac{1-\sigma}{\sigma}}\, dy.$$

The function $y^{\frac{1-\sigma}{\sigma}}$ is in $L^1([0,2\pi])$ (in fact, it is even continuous), so the last integral is the Fourier coefficient of this function evaluated at the frequency kaN^σ, which tends to $+\infty$ with N. By the Riemann-Lebesgue lemma this Fourier coefficient is $o(1)$ (tends to 0) so our integral divided by N is clearly $o(N)$. (Strictly speaking, the fact that the frequency kaN^σ is not an integer does not allow us to call this a "Fourier coefficient", but the Riemann-Lebesgue lemma still holds, with the same proof.)

Therefore it is enough to show that

$$|I_{k,N} - S_{k,N}| = o(N).$$

We have

$$|I_{k,N} - S_{k,N}| \le \sum_{n=1}^{N-1} \left| \int_n^{n+1} f_k(x)\, dx - f_k(n) \right|$$
$$= \sum_{n=1}^{N-1} \left| \int_n^{n+1} (f_k(x) - f_k(n))\, dx \right| dx$$

Using the mean value theorem on f_k we have

$$f_k(x) - f_k(n) = f_k'(\xi)(x-n), \text{ for some } \xi \in (n, x),$$

so

$$|f_k(x) - f_k(n)| \le |f_k'(\xi)| = 2\pi |ka| \sigma \frac{1}{\xi^{1-\sigma}} \le \frac{C}{n^{1-\sigma}}.$$

Substituting in the inequality above we get

$$|I_{k,N} - S_{k,N}| \le C \sum_{n=1}^{N-1} \frac{1}{n^{1-\sigma}} = O(N^\sigma) = o(N),$$

as we had to show.